Augenblick, Lazarus: Restrictions on Asset-Price Movements Under Rational Expectations: Theory and Evidence

Discussion by Jaroslav Borovička (NYU)
July 2018
1. Beliefs, risk adjustments and asset prices
   • information in no-arbitrage restrictions

2. Summary of the paper **methodology and results**
   • martingale restrictions on beliefs
   • localization of state prices
   • implied slopes of the SDF

3. Role of **assumptions**
   • key assumption: constant SDF slope

4. Questions and suggestions
   • are we testing for rational expectations?
High volatility in prices of risky securities

- high frequency (intraday, daily returns)
- business cycle frequency (monthly, quarterly returns)

Are these asset price fluctuations ‘rational’?

- i.e., can they be explained by a plausible model of investors’ preferences, market constraints, and investors’ beliefs that are correct (in line with the data-generating process (DGP))?
Abstract from market constraints, assume no arbitrage.

• then, for any investor’s subjective belief $\hat{\mathcal{P}}$, there exists an SDF $m_{t+1}$ that prices financial assets

$$1 = \hat{E}_t [m_{t+1}R_{t+1}].$$  (1)
Abstract from market constraints, assume no arbitrage.

- then, for any investor’s subjective belief $\hat{P}$, there exists an SDF $m_{t+1}$ that prices financial assets

$$1 = \hat{E}_t [m_{t+1} R_{t+1}] .$$  \hfill (1)

- but of course, defining $h = d\hat{P}/dP$, we can write

$$1 = E_t[\underbrace{m_{t+1} h_{t+1} R_{t+1}}_{\hat{m}_{t+1}}]$$ \hfill (2)
Abstract from market constraints, assume no arbitrage.

- then, for any investor’s subjective belief \( \hat{P} \), there exists an SDF \( m_{t+1} \) that prices financial assets

\[
1 = \hat{E}_t [m_{t+1} R_{t+1}] .
\]  

(1)

- but of course, defining \( h = d\hat{P}/dP \), we can write

\[
1 = E_t [\underbrace{m_{t+1} h_{t+1} R_{t+1}}_{\hat{m}_{t+1}}]
\]  

(2)

Which is the right model of the SDF, \( m_{t+1} \) or \( \hat{m}_{t+1} \)?

- that depends on what is plausible \( \implies \) impose structure on the SDF
- asset price data alone cannot tell (1) and (2) apart
  - even if we perfectly know the DGP measure \( P \)
Information from no-arbitrage restrictions

\[ 1 = E_t[m_{t+1} h_{t+1} R_{t+1}] \]

\[ \hat{m}_{t+1} \]

- Time series information identifies the DGP measure \( P \).
- Data on returns \( R_{t+1} \) together with no-arbitrage restrictions inform \( \hat{m}_{t+1} \).
- We must impose additional restrictions to separately infer something about \( m_{t+1} \) and \( h_{t+1} \).
This paper

\[ 1 = E_t[m_{t+1}h_{t+1}R_{t+1}] \]

\[ \hat{m}_{t+1} \]

• **imposes** the zero of **correct beliefs** \[ \Rightarrow h_{t+1} \equiv 1 \]

• restricts the class of SDFs (**conditional transition independence (CTI)**) \[ \Rightarrow \text{restriction on } m_{t+1} \]

• **uses martingale restrictions under DGP** (time series information) to deduce whether \( \exists m_{t+1} \) in the given class that also prices returns
This paper

\[ 1 = E_t [ \hat{m}_{t+1} h_{t+1} R_{t+1} ] \]

- imposes the zero of correct beliefs \( h_{t+1} = 1 \)
- restricts the class of SDFs (conditional transition independence (CTI)) \( \Rightarrow \) restriction on \( m_{t+1} \)
- uses martingale restrictions under DGP (time series information) to deduce whether \( \exists m_{t+1} \) in the given class that also prices returns

Outcomes

- if \( \exists \) such \( m_{t+1} \), characterize the minimum needed local risk aversion consistent with \( h_{t=1} = 1 \)
- if \( \nexists \) such \( m_{t+1} \), conclude that \( h_{t+1} \neq 1 \).
MARTINGALE RESTRICTIONS

Let $X_t$ be a square-integrable martingale. Then

$$
E_t \left[ \sum_{j=t}^{T-1} (X_{j+1} - X_j)^2 \right] = E_t \left[ X_T^2 - X_t^2 \right]
$$

• restriction on quadratic variation of a martingale
MARTINGALE RESTRICTIONS

Let $X_t$ be a square-integrable martingale. Then

$$E_t \left[ \sum_{j=t}^{T-1} (X_{j+1} - X_j)^2 \right] = E_t \left[ X_T^2 - X_t^2 \right]$$

- restriction on quadratic variation of a martingale

‘Beliefs are martingales’ (under their own measure)

- apply the above result to a two-state outcome $\implies$ localization
- $\pi_t$ — probability of the low state realization at $T$

$$E_t [m_T] \triangleq E_t \left[ \sum_{j=t}^{T-1} (\pi_{j+1} - \pi_j)^2 \right] = E_t \left[ (1 - \pi_t) \pi_t - (1 - \pi_T) \pi_T \right] \triangleq E_t [r_T]$$

- if we could measure beliefs, this would constitute a test of rational expectations (Augenblick and Rabin (2018))
- restriction on how much beliefs can vary over time
Instead, we observe risk-neutral (Arrow–Debreu) prices $\pi_t^*$. Then

$$E_t [m_T^*] \equiv E_t \left[ \sum_{j=t}^{T-1} (\pi_{j+1}^* - \pi_j^*)^2 \right] \neq E_t \left[ (1 - \pi_t^*) \pi_t^* - (1 - \pi_T^*) \pi_T^* \right] \equiv E_t [r_T^*]$$

Relationship between $\pi_t^*$ and $\pi_t$

$$\pi_t^* \propto E_t [M_T/M_t \mid s_T = \text{low}] \pi_t$$

• reflects risk adjustment and normalization by the risk-free rate
RESTRICTING THE SDF SLOPE

Rewrite relation between $\pi_t^*$ and $\pi_t$ in terms of odds ratio

$$\frac{\pi_t^*}{1 - \pi_t^*} = \frac{E_t [M_T / M_t | s_T = \text{low}]}{E_t [M_T / M_t | s_T = \text{high}]} \frac{\pi_t}{1 - \pi_t}$$

risk-neutral prices  
SDF slope $\phi$  
beliefs

If we observe (a lot of) variation in $\pi_t^*$ over time, what can we infer about variation in $\pi_t$?

- we must restrict movements in the SDF slope $\phi$ over time
- $\phi$ can differ across terminal state realizations
Assumption: Conditional transition independence (CTI)

- $\phi$ is constant over time (it can also follow a martingale)

\[
\frac{\pi_t^*}{1 - \pi_t^*} = \frac{E_t [M_T/M_t \mid s_T = \text{low}]}{E_t [M_T/M_t \mid s_T = \text{high}]} \frac{\pi_t}{1 - \pi_t}
\]

risk-neutral prices \hspace{5cm} SDF slope $\phi$ \hspace{5cm} beliefs

Idea: A high $\phi$ translates large variation in $\pi_t^*$ into modest variation in $\pi_t$, which can then satisfy the martingale restriction on correct beliefs.

\[
E_t [m_T^* - r_t^*] \leq (\pi_t^*)^2 \left(1 - \frac{1}{\pi_t^* + \phi (1 - \pi_t^*)}\right)
\]
Define states: Terminal values of a market index $R_T^m = s_T$ at some date $T$.

Extract Arrow–Debreu prices (Breeden, Litzenberger (1978))

Construct sample averages for $E_t [m_T^* - r_t^*]$

- daily, weekly, monthly frequencies
- conditional on terminal returns for various adjacent payoff states $R_T^m \in \{s_j, s_{j+1}\} \implies$ localization of the return distribution

Infer the required $\phi$’s to satisfy inequality

$$E_t [m_T^* - r_t^*] \leq (\pi_t^*)^2 \left(1 - \frac{1}{\pi_t^* + \phi(1 - \pi_t^*)}\right)$$
SUMMARY OF THE RESULTS

Required SDF slopes $\phi$ need to be very high to rationalize correct beliefs.

- Consequence of high volatility in $\pi_t^*$. 
Figure 5: Estimates of Relative Risk Aversion: Splits by Time to Expiration

Notes: Estimates are constructed using the bound in Proposition 4, converted to relative risk aversion values using Proposition 5. Within each time-to-expiration subset, each point shows estimate for state pair \((s_j, s_{j+1})\) plotted at the excess return at the midpoint of those two states, \((s_j + s_{j+1})/2\), where \(s_j\) and \(s_{j+1}\) are in the state space \(S_{alt} = \exp(\{(−\infty, −0.225), −0.175, −0.125, \ldots, 0.125, 0.175, (0.225, \infty)\})\). Each estimate in the \(\leq 2\) weeks series uses risk-neutral belief movement and uncertainty resolution observations from \(t_i = T_i - 10\) to \(t_i = T_i\), or \(m^* T_i - 10, T_i\), \(i, j\) and \(r^* T_i - 10, T_i\), \(i, j\), respectively, and similarly for the remainder of the series. Gray band indicates break in the \(y\)-axis. Estimates aggregated across all state pairs (excluding the extreme state pairs \((s_1, s_2)\) and \((s_{J−1}, s_J)\)) for each series: \(\hat{\gamma}_j = 0\) for \(\leq 2\) weeks (95% confidence interval: \([0, \infty)\)); \(\hat{\gamma}_j = 250\) for 3–4 weeks (CI: \([170, \infty)\)); \(\hat{\gamma}_j = \infty\) for 5–8 weeks (CI: \([3980, \infty)\)); \(\hat{\gamma}_j = \infty\) for \(\geq 9\) weeks (CI: \(\infty\)). CIs calculated using block bootstrap with blocks of 45 days, 5,000 draws.

As we move up the chart, each successive series of risk-aversion estimates for more-distant horizons from expiration is weakly greater than the preceding series at every point in the return distribution. The estimates using data for trading dates between three and four weeks from expiration (blue circles) are large or infinite for negative excess-return states, and around \(\hat{\gamma}_j = 50\) for positive states; using data from the second-to-last month from expiration (light blue diamonds) yields slightly greater estimates exhibiting a similar pattern; and using data from trading dates more than two months from expiration (green squares) yields \(\hat{\gamma}_j = \infty\) for almost all return states.

The pooled estimates are \(\hat{\gamma}_j = 250\) for the three-to-four-week data (with one-sided 95 percent confidence interval lower bound of 170), and \(\hat{\gamma}_j = \infty\) for the last two sets of data (with confidence interval lower bounds of 3,980 and \(\infty\), respectively). Each successive pooled estimate is also significantly greater than the previous value at the 95 percent level, with the exception of the last two.
Required SDF slopes $\phi$ need to be very high to rationalize correct beliefs.

- Consequence of high volatility in $\pi_t^*$.

Higher $\phi$ needed to rationalize higher frequency (daily) variation.

- Is there more ‘unexplained’ variation at higher frequencies? Unclear, only a necessary bound.
Figure 6: Estimates of Relative Risk Aversion: Splits by Sampling Frequency

Notes:
Estimates are constructed using the bound in Proposition 4, converted to relative risk aversion values using Proposition 5. Within each sampling-frequency subset, each point shows estimate for state pair $(s_j, s_{j+1})$ plotted at the excess return at the midpoint of those two states, $(s_j + s_{j+1})/2$, where $s_j$ and $s_{j+1}$ are in the state space $S_{\text{baseline}} = \exp\left(\{-\infty, -0.11, -0.09, -0.07, \ldots, 0.07, 0.09, (0.11, \infty)\}\right)$. Each estimate in the monthly series uses risk-neutral beliefs $\tilde{\pi}_t^*, i, j$ sampled only on the second Wednesday of each month; the biweekly series samples beliefs on the Wednesdays of the evenly-numbered weeks of the year; the weekly series samples beliefs every Wednesday; and the daily series every trading day. Gray band indicates break in the $y$-axis. Estimates aggregated across all state pairs (excluding the extreme state pairs $(s_1, s_2)$ and $(s_{J-1}, s_J)$) for each series: $\hat{\gamma} = 123$ for monthly (95% confidence interval: [97, $\infty$)); $\hat{\gamma} = 425$ for biweekly (CI: [228, $\infty$]); $\hat{\gamma} = \infty$ for weekly (CI: [4950, $\infty$]); $\hat{\gamma} = \infty$ for daily (CI: $\infty$). CIs calculated using block bootstrap with blocks of 45 days, 5,000 draws.

We thus find that the monthly belief variation can be explained with finite but nonetheless still quite large risk-aversion values. Further, this monthly variation masks additional volatility and required risk aversion at higher sampling frequencies, and so any RE model capable of matching the moments of the risk-neutral belief process at a monthly horizon would seem to possess what might be thought of as incorrect statistical microfoundations.

There are, however, two additional possibilities that may account for the differences between the estimates for the more- versus less-frequently-sampled series. First, it is possible that the greater measured belief volatility at higher sampling frequencies is a result of measurement error, as Figure 6 again does not account for such noise. But in this case, the fact that required SDF slopes $\phi$ need to be very high to rationalize correct beliefs.

- Consequence of high volatility in $t$.
- Higher $\phi$ needed to rationalize higher frequency (daily) variation.
- Is there more 'unexplained' variation at higher frequencies? Unclear, only a necessary bound.
- High risk aversion needed across all terminal payoff states.
- Informative about the type of risks faced by investors (e.g., tail risk).
Required SDF slopes $\phi$ need to be very high to rationalize correct beliefs.

- Consequence of high volatility in $\pi_t^*$.  

**Higher $\phi$ needed to rationalize higher frequency (daily) variation.**

- Is there more ‘unexplained’ variation at higher frequencies? Unclear, only a necessary bound.

High risk aversion needed **across all terminal payoff states.**

- Informative about the type of risks faced by investors (e.g., tail risk).
Figure 6: Estimates of Relative Risk Aversion: Splits by Sampling Frequency

Notes:
Estimates are constructed using the bound in Proposition 4, converted to relative risk aversion values using Proposition 5. Within each sampling-frequency subset, each point shows estimate for state pair \((s_j, s_{j+1})\) plotted at the excess return at the midpoint of those two states, \((s_j + s_{j+1})/2\), where \(s_j\) and \(s_{j+1}\) are in the state space \(S_{\text{baseline}} = \exp(\{-\infty, -0.11, -0.09, -0.07, \ldots, 0.07, 0.09, (0.11, \infty)\})\). Each estimate in the monthly series uses risk-neutral beliefs \(\tilde{\pi}^{*}_t, i, j\) sampled only on the second Wednesday of each month; the biweekly series samples beliefs on the Wednesdays of the evenly-numbered weeks of the year; the weekly series samples beliefs every Wednesday; and the daily series every trading day. Gray band indicates break in the \(y\)-axis. Estimates aggregated across all state pairs (excluding the extreme state pairs \((s_1, s_2)\) and \((s_{J-1}, s_J)\)) for each series: \(\hat{\gamma}_j = 123\) for monthly (95% confidence interval: [97, \(\infty\)));
\(\hat{\gamma}_j = 425\) for biweekly (CI: [228, \(\infty\)));
\(\hat{\gamma}_j = \infty\) for weekly (CI: [4950, \(\infty\)));
\(\hat{\gamma}_j = \infty\) for daily (CI: [\(\infty\)). CIs calculated using block bootstrap with blocks of 45 days, 5,000 draws.

Which are all mutually statistically different at the 95% level aside from the daily versus weekly estimates. The lower bound of the confidence interval for monthly data is 97. Separately, the return-state-specific monthly estimates also exhibit a roughly decreasing pattern across possible return values.

We thus find that the monthly belief variation can be explained with finite but nonetheless still quite large risk-aversion values. Further, this monthly variation masks additional volatility and required risk aversion at higher sampling frequencies, and so any RE model capable of matching the moments of the risk-neutral belief process at a monthly horizon would seem to possess what might be thought of as incorrect statistical microfoundations.

There are, however, two additional possibilities that may account for the differences between the estimates for the more- versus less-frequently-sampled series. First, it is possible that the greater measured belief volatility at higher sampling frequencies is a result of measurement error, as Figure 6 again does not account for such noise. But in this case, the fact that required

\[\phi\]

SDF slopes need to be very high to rationalize correct beliefs.

• Consequence of high volatility in \(t\).

Higher \(\phi\) needed to rationalize higher frequency (daily) variation.

• Is there more 'unexplained' variation at higher frequencies? Unclear, only a necessary bound.

High risk aversion needed across all terminal payoff states.

• Informative about the type of risks faced by investors (e.g., tail risk).
DISCUSSION OF ASSUMPTIONS

The whole identification hinges on the choice of class of ‘plausible’ SDFs.

\[
\frac{\pi_t^*}{1 - \pi_t^*} = \frac{E_t \left[ M_T/M_t \mid R_T^m = \text{low} \right]}{E_t \left[ M_T/M_t \mid R_T^m = \text{high} \right]} \frac{\pi_t}{1 - \pi_t} \]

constant SDF slope \( \phi \)

\[
\phi = \frac{u'(R_T^m = \text{low})}{u'(R_T^m = \text{high})}
\]

To first-order approximation, this corresponds to constant local relative risk aversion \( \gamma (\bar{R}) \)

- market risk premium

\[
E_t \left[ R_T^e \right] = \gamma (\bar{R}) \ Var_t \left[ R_T^e \right]
\]

- \( \gamma (\bar{R}) \) is the local curvature of the utility/value function as a function of the return realization, by assumption independent of \( t \)

- time-variation in market risk premium can only be driven by \( Var_t \left[ R_T^e \right] \)
WHEN DOES THE ASSUMPTION HOLD?

Separable utility over returns at terminal state

- **caveat:** if utility is over consumption, then time variation in the mapping between returns and consumption can lead to violations

Epstein–Zin in iid growth environment

- not very interesting, isomorphic to recalibrated separable CRRA utility

Disaster risk model (Gabaix (2012)) with CRRA utility

- holds for (high) return realizations conditional on which disasters between today and maturity have a negligible chance
- **interesting:** utilizes localization to return dynamics of particular AD securities
- for a rejection of rationality, it requires establishing what negligible chance means in the model and data

In general, $\phi$ cannot move systematically with $\pi_t^*$. 
DEVIATIONS FROM CONSTANT RISK AVERSION

\[
\frac{\pi^*_t}{1 - \pi^*_t} = \frac{E_t[M_T/M_t \mid R^m_T = \text{low}]}{E_t[M_T/M_t \mid R^m_T = \text{high}]} \cdot \frac{\pi_t}{1 - \pi_t}
\]

constant SDF slope \( \phi \)

Inference with constant \( \phi \)

- bad times \( \implies \) observed \( \uparrow \pi^*_t \) \( \implies \) infer required increase in \( \pi_t \), given \( \phi \)

Systematic variation in \( \phi \)

- bad times \( \implies \) observed \( \uparrow \pi^*_t \) \( \implies \) associated increase in \( \phi \) \( \implies \) lower required increase in \( \pi_t \)

- lower average \( \phi \) needed to dampen implied variation in \( \pi_t \) sufficiently to satisfy the martingale restriction
**WHAT IS OFF THE TABLE?**

- **Nonseparable preferences** (Epstein–Zin) with interesting state dynamics
- Explicit *time-variation in risk aversion*
- Models with agent heterogeneity (e.g., in risk aversion)
  - implied risk aversion a function of wealth distribution
  - *how to interpret results in models with heterogeneous beliefs?*
- Models with *financial constraints*
  - implied risk aversion changes with tightness of constraints
- Habit formation models *(Campbell and Cochrane (1999))*
  - implied risk aversion function of habit level
  - authors show that in CC (1999), the bound still holds approximately
    - but SDF slopes implied by the model are trivial for non-extreme states
A REINTERPRETATION OF THE PROBLEM

\[
\frac{\pi_t^*}{1 - \pi_t^*} = \frac{E_t [M_T/M_t \mid R_T^m = \text{low}]}{E_t [M_T/M_t \mid R_T^m = \text{high}]} \left\{ \begin{array}{l}
\frac{\pi_t}{1 - \pi_t}
\end{array} \right\}
\]

\text{time-varying SDF slope } \phi_t

**Question:** How much variation in \( \phi_t \) is needed such that implied \( \pi_t \) satisfy the martingale restriction?

- a test of a larger class of (more interesting) models

Existing models may well still fail

- many asset pricing models designed to match unconditional risk premia and the cross-section
- appropriate time-variation harder to achieve
OTHER CONSIDERATIONS

Implementation relies purely on option prices

• studies option price dynamics (with all their specifics) relative to the index
• what about correct beliefs in the pricing of the index itself?

Authors relate identified excess belief movement to macro statistics

• why not survey data directly?

Why bounds at all?

• Given the information obtained from localization, one could directly attempt to estimate the SDF nonparametrically (Christensen (2017))

Approximation details (discretization, etc.)

• authors take these seriously, but details are subtle

Localization

• we do not need to test adjacent states only, additional information in alternative state combinations
Given stated assumptions, the paper is cleanly executed, with a lot of detail.

- robustness checks, careful econometrics
- models could be studied in more detail

Interesting results

- much higher high frequency (daily) variation than variation at ‘macro’ frequencies
  - Market microstructure effects in daily data? Different models needed?
  - Caution regarding bid-ask bounce? (not covered here)
- localization of return dynamics can discard some models
  - high AD price variation for high-return states speaks against disaster-type models that focus risk adjustments to adverse states
A test of rational expectations?

- restricts attention to a class of models that is too narrow
- reverting the problem and studying required variation in implied risk aversion may be more fruitful

Even if we impose stringent restrictions on SDF and reject correct beliefs, the puzzle still stands.

- why do beliefs fluctuate so much?
- need to devise interesting models of belief dynamics to make progress ...
- ... just like interesting models of risk adjustments that go beyond imposed restrictions ...
- ... and in some cases, the two cannot be distinguished using asset price data anyway
  - e.g. Epstein–Zin with unitary IES vs. Hansen–Sargent robust preference vs. ex-post Bayesian worst-case belief
Ross (2015) ‘Recovery Theorem’

- uses only cross-sectional price information, no time series
- imposes separable marginal utility (transition independence (TI))
  \[ m_{t+1} = \beta \frac{u'(s_{t+1})}{u'(s_t)} \]
- and stationarity of the state dynamics
- under TI, it recovers subjective belief \( \hat{P} \) (possibly different from \( P \))
- if TI does not hold, it recovers a long-run risk neutral measure (different from \( \hat{P} \) or \( P \))

This paper

- imposes a restriction based on time-series information in \( P \) (time-variation in prices)
- asks whether imposing \( P = \hat{P} \) can be consistent with SDFs that satisfy CTI
- allows for nonstationarity of the state and hence for some martingale components in SDF
- both papers assume time-invariant local curvature over terminal states