Bakshi, Gao, Panayotov: A theory of dissimilarity between stochastic discount factors

Discussion by Jaroslav Borovička (NYU)
January 2018
Construct a discrepancy measure — the **Hellinger distance** $H(m, m^*)$ — that

- has some appealing properties
- can be inferred from asset price data

Provide empirical evidence on $H(m, m^*)$

- from the cross section of currency option prices
- from the time series of currency returns
- from estimated SDFs using ‘model-free’ restrictions
- from structural models in international finance

Metrics allowing comparisons across classes of models are important.
1. Are the appealing properties utilized effectively?
2. Does the Hellinger distance provide insights different from other discrepancy measures?
3. Are quantitative implications clearly spelled out?
4. Have we learned something new?
5. Are there alternative interpretations of the discrepancy measure?
No-arbitrage condition in complete international markets

\[ m_{t+1} \frac{S_{t+1}}{S_t} = m_{t+1}^* \]

- \( m_{t+1}, m_{t+1}^* \) domestic and foreign one-period SDFs
  - assume throughout that these are normalized to have conditional mean 1
- \( S_{t+1}/S_t \) exchange rate depreciation
Hellinger distance as a measure of dissimilarity of \( m_{t+1} \) and \( m^*_{t+1} \)

\[
H (m_{t+1}, m^*_{t+1}) = 1 - E_t \left[ \sqrt{m_{t+1} m^*_{t+1}} \right]
\]

- see Schneider and Trojani (2015), Schneider (2017) for more extensive applications of Hellinger divergence in the study of market returns

A simple manipulation yields

\[
H (m_{t+1}, m^*_{t+1}) = 1 - E_t \left[ m_{t+1} \sqrt{\frac{m^*_{t+1}}{m_{t+1}}} \right] = 1 - E_t^Q \left[ \sqrt{\frac{S_{t+1}}{S_t}} \right]
\]

- RHS can be inferred from a cross-section of currency option prices
Under log-normality of $m_{t+1}$ and $m_{t+1}^*$

$$H(m_{t+1}, m_{t+1}^*) = 1 - \exp\left(-\frac{1}{8} \text{Var}_t \log \left( \frac{S_{t+1}}{S_t} \right) \right)$$

• when increments are independent, $\text{Var}_t [\cdot]$ can be proxied with daily variation

Use the two approaches to infer nonnormalities in SDFs.
RESULT: EVIDENCE OF NON-NORMALITIES

<table>
<thead>
<tr>
<th></th>
<th>√(H \times 100) Mean</th>
<th>Std.</th>
<th>Min.</th>
<th>Max.</th>
<th>5th</th>
<th>25th</th>
<th>50th</th>
<th>75th</th>
<th>95th</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Hellinger measures based on currency option prices</td>
<td>(H_t^{[45]})</td>
<td>1.14</td>
<td>0.33</td>
<td>0.64</td>
<td>3.39</td>
<td>0.80</td>
<td>0.92</td>
<td>1.09</td>
<td>1.26</td>
</tr>
<tr>
<td></td>
<td>(H_t^{US,[9]})</td>
<td>1.12</td>
<td>0.34</td>
<td>0.59</td>
<td>3.34</td>
<td>0.73</td>
<td>0.94</td>
<td>1.08</td>
<td>1.24</td>
</tr>
<tr>
<td>B. Hellinger measures based on variance of currency returns</td>
<td>(H_t^{[45]})</td>
<td>1.05</td>
<td>0.36</td>
<td>0.55</td>
<td>3.90</td>
<td>0.69</td>
<td>0.84</td>
<td>0.99</td>
<td>1.13</td>
</tr>
<tr>
<td></td>
<td>(H_t^{US,[9]})</td>
<td>1.03</td>
<td>0.37</td>
<td>0.48</td>
<td>3.53</td>
<td>0.61</td>
<td>0.83</td>
<td>0.97</td>
<td>1.15</td>
</tr>
<tr>
<td>C. Deviations</td>
<td>(\log(\frac{H_t^{[45]} / H_t^{[45]}}{H_t^{[45]} / H_t^{[45]}})), %</td>
<td>10.3</td>
<td>12.8</td>
<td>-19.4</td>
<td>47.0</td>
<td>-10.5</td>
<td>1.2</td>
<td>10.0</td>
<td>18.2</td>
</tr>
<tr>
<td></td>
<td>(\log(\frac{H_t^{US,[9]} / H_t^{US,[9]}}{H_t^{US,[9]} / H_t^{US,[9]}})), %</td>
<td>11.1</td>
<td>14.4</td>
<td>-19.7</td>
<td>53.0</td>
<td>-10.0</td>
<td>1.1</td>
<td>10.5</td>
<td>19.9</td>
</tr>
</tbody>
</table>

- “We interpret the time-varying nature of the deviations as the contribution of stochastically-varying risk-neutral moments of currency returns”
- “Evidence of dissimilarity in higher moments of SDFs”
Evidence of non-normality or noise?
Why is the Hellinger distance a superior measure of non-normality in currency returns?
Why not study the distribution of $S_{t+1}/S_t = m^*_t/m_{t+1}$ directly?
The Hellinger measure is computed as in equation (17):

$$H_t = \sqrt{R_{f,t+1}} \cdot \frac{1}{16} F_t \left( \int \{ K > F_t \} \cdot C_t[K] \left( K^3 = 2 d_K \right) + \int \{ K < F_t \} \cdot P_t[K] \left( K^3 = 2 d_K \right) \right)$$

where $C_t[K] \left( P_t[K] \right)$ is the price of a call (put) on the foreign exchange with strike price $K$, $F_t$ is the forward exchange rate, and $R_{f,t+1}$ is the gross return on the domestic risk-free bond. We define the following:

- $H_{t,[i,j]} \equiv \sqrt{H_{t}}$ (based on monthly data, in %) for economy pair $(i,j)$ in month $t$;
- $H_{i,[9]} t \equiv \text{Cross-sectional average: } \frac{1}{9} \sum_{j=1}^{9} H_{t,[i,j]}$ for economy $i$ in month $t$; and
- $H_{[45]} t \equiv \text{Cross-sectional average of } H_{t,[i,j]}$ for all 45 pairs of economies in month $t$.

The first column in the table shows the mean difference between each of the economy-specific average measures $H_{i,[9]} t$ and $H_{[45]} t$. The remaining columns show 95% confidence intervals for these mean differences, obtained with 10,000 stationary bootstrap samples, as well as the respective minimum and maximum of the bootstrapped mean differences. The sample period is from 1/1996 to 6/2014 (and from 1/1999 for pairs including NO or EU). All numbers are reported in percent.

<table>
<thead>
<tr>
<th>Country</th>
<th>Mean</th>
<th>[95% CI]</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euro-zone (EU)</td>
<td>-0.17</td>
<td>[-0.21, -0.13]</td>
<td>-0.26</td>
<td>-0.12</td>
</tr>
<tr>
<td>United Kingdom (UK)</td>
<td>-0.09</td>
<td>[-0.10, -0.07]</td>
<td>-0.11</td>
<td>-0.05</td>
</tr>
<tr>
<td>Switzerland (SW)</td>
<td>-0.07</td>
<td>[-0.11, -0.03]</td>
<td>-0.14</td>
<td>0.02</td>
</tr>
<tr>
<td>Norway (NO)</td>
<td>-0.05</td>
<td>[-0.08, -0.03]</td>
<td>-0.11</td>
<td>0.00</td>
</tr>
<tr>
<td>Sweden (SD)</td>
<td>-0.03</td>
<td>[-0.06, -0.01]</td>
<td>-0.08</td>
<td>0.01</td>
</tr>
<tr>
<td>United States (US)</td>
<td>-0.01</td>
<td>[-0.08, 0.05]</td>
<td>-0.15</td>
<td>0.09</td>
</tr>
<tr>
<td>Canada (CA)</td>
<td>0.00</td>
<td>[-0.04, 0.04]</td>
<td>-0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>Australia (AU)</td>
<td>0.04</td>
<td>[0.01, 0.07]</td>
<td>-0.01</td>
<td>0.11</td>
</tr>
<tr>
<td>New Zealand (NZ)</td>
<td>0.12</td>
<td>[0.08, 0.17]</td>
<td>0.05</td>
<td>0.22</td>
</tr>
<tr>
<td>Japan (JP)</td>
<td>0.23</td>
<td>[0.16, 0.32]</td>
<td>0.09</td>
<td>0.43</td>
</tr>
</tbody>
</table>

- Are these differences quantitatively large?
- Interpretation of the magnitude of the Hellinger distance?
Infer $m_{t+1}$ (and $m_{t+1}^*$) from a minimum discrepancy problem

$$\min_{m_{t+1}} E_t [m_{t+1} \log m_{t+1}]$$

subject to

$$1 = E_t [m_{t+1} R_{t+1}]$$
$$1 = E_t \left[ m_{t+1} \frac{S_{t+1}}{S_t} R_{t+1}^* \right]$$

- Well-studied problem in the class of Cressie and Read (1984) discrepancies

Compare inferred $m_{t+1}^*/m_{t+1}$ across countries with previous results

- use the Hellinger distance and other divergence measures
### Table 4: Hellinger and Chi-squared measures from minimum discrepancy problems

First we solve the optimization (minimum discrepancy) problems in equations (40) and (41), with the U.S. as the domestic economy and each of seven other economies as the foreign one. Then we use the time series length in months, and all samples end in 12/2016.

<table>
<thead>
<tr>
<th>Country</th>
<th>$T$</th>
<th>Hellinger measure</th>
<th>Chi-squared measure</th>
<th>Volatility $\tilde{n}$</th>
<th>$\tilde{n}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EU</td>
<td>191</td>
<td>0.81</td>
<td>0.096</td>
<td>105</td>
<td>104</td>
</tr>
<tr>
<td>UK</td>
<td>336</td>
<td>0.74</td>
<td>0.081</td>
<td>77</td>
<td>76</td>
</tr>
<tr>
<td>SW</td>
<td>336</td>
<td>0.89</td>
<td>0.107</td>
<td>84</td>
<td>84</td>
</tr>
<tr>
<td>CA</td>
<td>319</td>
<td>0.62</td>
<td>0.056</td>
<td>85</td>
<td>84</td>
</tr>
<tr>
<td>AU</td>
<td>336</td>
<td>0.97</td>
<td>0.127</td>
<td>74</td>
<td>70</td>
</tr>
<tr>
<td>NZ</td>
<td>309</td>
<td>0.98</td>
<td>0.138</td>
<td>76</td>
<td>70</td>
</tr>
<tr>
<td>JP</td>
<td>336</td>
<td>0.91</td>
<td>0.113</td>
<td>92</td>
<td>95</td>
</tr>
</tbody>
</table>

- Are the results ‘close’ to $H(m, m^*)$ obtained earlier?
- Hellinger and chi-squared measure are highly correlated.
- Why prefer the Hellinger measure?
  - Some computational simplicity but other than that?
## APPROACH 4: MODEL COMPARISON

### A) Verdelhan (2010); B) Lustig, Roussanov, Verdelhan (2014); C) Colacito, Croce (2011); D) time-varying disasters

### Table 5

<table>
<thead>
<tr>
<th>Models</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H_t^{[45]}$</td>
</tr>
<tr>
<td>A (i)</td>
<td>1.14</td>
</tr>
<tr>
<td>B (ii)</td>
<td>-</td>
</tr>
<tr>
<td>C (iii)</td>
<td>1.30</td>
</tr>
</tbody>
</table>

### Caveats
- Models differ in $H(m, m^*)$
  - Caveats: Calibrations to different periods, ...
- How much is economically significant?
- Should we calibrate models specifically to Hellinger distance?
  - What value added relative to other moments?
What if there is a friction in international financial markets?

\[
m_{t+1} \frac{S_{t+1}}{S_t} \frac{F_{t+1}}{F_t} = m_{t+1}^*
\]

- \(F_{t+1}/F_t\) represents evolution of shadow prices on financial constraints

Hellinger distance computed from currency options

\[
\frac{S_{t+1}}{S_t} = \frac{m_{t+1}^*}{m_{t+1}} \frac{F_t}{F_{t+1}}
\]

does not measure only the discrepancy in SDFs.

- but different ways of inferring \(m_{t+1}^*/m_{t+1}\) could inform us about \(F_{t+1}/F_t\)
Creative, interesting paper

- model-free bounds have been informative about key restrictions on SDFs (Hansen and Jagannathan (1991))

Improve interpretation of the results

- which distribution characteristics does the Hellinger distance accentuate?
- quantitative interpretation of the distance

Can we utilize alternative approaches to think about frictions in international markets?