A Theory of Rational Short-Termism with Uncertain Betas

Christian Gollier

Discussed by Jaroslav Borovička (NYU)

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Aim of the paper

The model
- iid consumption growth: $\Delta c_{t+1} = \mu_g + \sigma_g w_{t+1}$
- pricing of levered consumption claims: $c_t^\beta$
- uncertainty about $\beta$.

The question
- Implication of $\beta$ uncertainty on discounting of long-term claims
  - long-term investment projects, environmental assets, social security, . . .

The answer
- For plausible parameterizations, long-term assets discounted at higher rates due to uncertainty in $\beta$.
- mean grows with $\beta t$, volatility with $\beta \sqrt{t}$, higher $\beta$ always dominates for large $t$. 
Temporal decomposition of risk

Payoffs at different horizons discounted at different rates

- zero coupon bond yield curve — risk-free payoffs
- temporal decomposition of risk

Empirical

- equity yields — Van Binsbergen, Hueskes, Koijen, Vrugt, and others
- some evidence in favor of decreasing risk premia

Theoretical

Shock elasticities

Risk premia reflect
- exposure of cash flows to risk
- price of this exposure (compensation for a unit of risk)

Shock exposure elasticity
- sensitivity of expected cash flow

Shock-price elasticity
- sensitivity of expected return
Expected returns and shock elasticities

- Expected return on cash flow $F_t$
  \[
  \frac{E[F_t]}{E[S_tF_t]} = \frac{E[E[F_t|\beta]]}{E[E[S_tF_t|\beta]]}
  \]

- Payoff $\beta$ perturbation $\log H(r) = rW$
  \[
  \beta(F_tH(r)) = \beta + rW
  \]

- Sensitivity of expected returns
  \[
  \begin{align*}
  \text{shock-price elasticity} & = \left. \frac{d}{dr} \log E[F_tH(r)] \right|_{r=0} \\
  \text{shock-exposure elasticity} & = \left. \frac{d}{dr} \log E[S_tF_tH(r)] \right|_{r=0}
  \end{align*}
  \]
Shock elasticities

**Experiment:** A marginal increase in the dispersion of $\beta$

- consequence for expected payoffs (shock-exposure elasticity)

$$
\left(1 - \sigma_g^2 \sigma_\beta^2 t\right)^{-1} \left(\mu_g + \sigma_g^2 \mu_\beta\right) \sigma_\beta
$$

- “correlated-risk-trend effect”

- consequence for expected returns (shock-price elasticity)

$$
\left(1 - \sigma_g^2 \sigma_\beta^2 t\right)^{-1} \gamma \sigma_g^2 \sigma_\beta
$$

Required compensations for additional risk diverge to $+\infty$.

- valuations **rise** with more risk / higher $\beta$ (for typical parametrizations)
- but expected payoffs rise even more
Discussion

Suitability of the endowment economy assumption

- If expected returns are so high (infinite), why investors don’t invest?
- They must dislike this type of risk.

What are the important sources of (parameter) uncertainty?

- long-run risk
- disasters

Resolving the uncertainty in $\beta$ over time

- interaction with nonseparable recursive preferences
- Collin-Dufresne, Johannes, Lochstoer (2013)
Conditional expectations

- Let $\beta = \mu_\beta + \sigma_\beta W$ where $W \sim N(0, 1)$. Then
  \[
  \exp \left( a\beta + b\beta^2 + c \right) = \exp \left( a\mu_\beta + b\mu_\beta^2 + c + AW + BW^2 \right)
  \]
  with
  \[A = a\sigma_\beta + 2b\mu_\beta \sigma_\beta \quad B = b\sigma_\beta^2\]

- Further
  \[E \left[ \exp \left( AW + BW^2 \right) f(W) \right] = k\tilde{E}[f(W)]\]
  with
  \[k = (1 - 2B)^{-1/2} \exp \left( \frac{1}{2} \frac{A^2}{1 - 2B} \right)\]
  and $\tilde{E}$ is the expectation under the distribution of $W$:
  \[W \sim N \left( (1 - 2B)^{-1} A, (1 - 2B)^{-1} \right)\]
Expected returns

- Expected returns are \((C_0 = 1)\)

\[
\frac{E[F_t]}{E[S_tF_t]} = \frac{E \left[ E \left[ C_t^\beta | \beta \right] \right]}{E \left[ E \left[ e^{-\delta t} C_t^{\beta-\gamma} | \beta \right] \right]} = \frac{E \left[ \exp \left( \mu_g t \beta + \frac{1}{2} \sigma_g^2 t \beta^2 \right) \right]}{\left[ \exp \left( -\delta t + \mu_g t (\beta - \gamma) + \frac{1}{2} \sigma_g^2 t (\beta - \gamma)^2 \right) \right]}
\]

- Expected payoff

\[
\begin{align*}
    a_t &= \mu_g t \\
    b_t &= \frac{1}{2} \sigma_g^2 t \\
    c_t &= 0 \\
    A_t^e &= a_t \sigma_\beta + 2 b_t \mu_\beta \sigma_\beta = \left( \mu_g + \sigma_g^2 \mu_\beta \right) \sigma_\beta t \\
    B_t^e &= b_t \sigma_\beta^2 = \frac{1}{2} \sigma_g^2 \sigma_\beta^2 t
\end{align*}
\]

so we get

\[
\exp \left( a_t \mu_\beta + b_t \mu_\beta^2 + c_t \right) (1 - 2B_t^e)^{-1/2} \exp \left( \frac{1}{2} \frac{A_t^2}{1 - 2B_t} \right) =
\]

\[
= \exp \left( \mu_g \mu_\beta t + \frac{1}{2} \sigma_g^2 \mu_\beta^2 t \right) \left( 1 - \sigma_g^2 \sigma_\beta^2 t \right)^{-1/2} \exp \left( \frac{1}{2} \frac{\left[ \left( \mu_g + \sigma_g^2 \mu_\beta \right) \sigma_\beta t \right]^2}{1 - \sigma_g^2 \sigma_\beta^2 t} \right)
\]
Expected returns

- Price

\[
a_t = \left( \mu_g - \gamma \sigma_g^2 \right) t \quad b_t = \frac{1}{2} \sigma_g^2 t \quad c_t = \left( -\delta - \gamma \mu_g + \frac{1}{2} \gamma^2 \sigma_g^2 \right) t
\]

\[
A_t^e = a_t \sigma_\beta + 2b_t \mu_\beta \sigma_\beta = \left( \mu_g - \gamma \sigma_g^2 + \sigma_g^2 \mu_\beta \right) \sigma_\beta t
\]

\[
B_t^e = b_t \sigma_\beta^2 = \frac{1}{2} \sigma_g^2 \sigma_\beta^2 t
\]

- so we get

\[
\exp \left( a_t \mu_\beta + b_t \mu_\beta^2 + c_t \right) \left( 1 - 2B_t^e \right)^{-1/2} \exp \left( \frac{A_t^2}{2 \left( 1 - 2B_t^e \right)} \right) =
\]

\[
= \exp \left( \left( \mu_g - \gamma \sigma_g^2 \right) \mu_\beta t + \frac{1}{2} \sigma_g^2 \mu_\beta^2 t + \left( -\delta - \gamma \mu_g + \frac{1}{2} \gamma^2 \sigma_g^2 \right) t \right) \cdot
\]

\[
\cdot \left( 1 - \sigma_g^2 \sigma_\beta^2 t \right)^{-1/2} \exp \left( \frac{1}{2} \left[ \left( \mu_g - \gamma \sigma_g^2 + \sigma_g^2 \mu_\beta \right) \sigma_\beta t \right]^2 \right)
\]

- The question is what explodes faster. Expected payoff explodes faster iff

\[
\left[ \left( \mu_g + \sigma_g^2 \mu_\beta \right) \sigma_\beta t \right]^2 > \left[ \left( \mu_g - \gamma \sigma_g^2 + \sigma_g^2 \mu_\beta \right) \sigma_\beta t \right]^2
\]

\[
\mu_\beta > \frac{1}{2} \gamma - \frac{\mu_g}{\sigma_g^2}
\]
Shock elasticities

- Consider a perturbation $\log H(r) = rW$. Consider

$$M_t = \exp \left( A_t W + B_t W^2 \right)$$

Then the shock elasticity for $M$ is

$$\frac{d}{dr} \log E[M_t H(r)] \bigg|_{r=0} = \frac{E[M_t W]}{E[M_t]} = \tilde{E}[W] = (1 - 2B_t)^{-1} A_t$$

- Shock-exposure elasticity for $C_t^\beta$

$$a_t = \mu_g t \quad b = \frac{1}{2} \sigma^2 t$$

$$A_t^e = a_t \sigma_\beta + 2b_t \mu_\beta \sigma_\beta = \left( \mu_g + \sigma^2 \mu_\beta \right) \sigma_\beta t$$

$$B_t^e = b_t \sigma^2_\beta = \frac{1}{2} \sigma^2 \sigma^2_\beta t$$

So the shock elasticities cease to exist for $t > (\sigma^2 \sigma^2_\beta)^{-1}$.

- The elasticity is

$$\left( 1 - \sigma^2_\beta t \right)^{-1} \left( \mu_g + \sigma^2 \mu_\beta \right) \sigma_\beta t$$
Shock elasticities

- **Shock-value elasticity for** $C_t^\beta$, i.e., shock elasticity for $S_t C_t^\beta = e^{-\delta t} C_t^{\beta-\gamma}$
  
  $$
a_t = \left(\mu_g - \gamma \sigma_g^2\right) t \quad b = \frac{1}{2} \sigma_g^2 t
  
  A_t^\nu = a_t \sigma_\beta + 2b_t \mu_\beta \sigma_\beta = \left(\mu_g - \gamma \sigma_g^2 + \sigma_g^2 \mu_\beta\right) \sigma_\beta t
  
  B_t^\nu = b_t \sigma_\beta^2 = \frac{1}{2} \sigma_g^2 \sigma_\beta^2 t
  $$

  So the shock elasticities ceases to exist for $t > \left(\sigma_g^2 \sigma_\beta^2\right)^{-1}$ as well

- **Shock-price elasticity**
  
  $$
  \left(1 - 2B_t^e\right)^{-1} A_t^e - \left(1 - 2B_t^\nu\right)^{-1} A_t^\nu = \left(1 - 2B_t^e\right)^{-1} \left(A_t^e - A_t^\nu\right) = \left(1 - \sigma_g^2 \sigma_\beta^2 t\right)^{-1} \gamma \sigma_g^2 \sigma_\beta t
  $$

- **Note**: because all the above elasticities assume an increase in exposure ($\beta$) in all periods, it is appropriate to rescale them by $t$. 