Stability of Equilibrium Asset Pricing Models: A Necessary and Sufficient Condition

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ABSTRACT. We obtain an exact necessary and sufficient condition for the existence and uniqueness of equilibrium asset prices in infinite horizon, discrete-time, arbitrage free environments. Using local spectral radius methods, we connect the condition, and hence the problem of existence and uniqueness of asset prices, with the recent literature on stochastic discount factor decompositions. Our results include a globally convergent method for computing prices whenever they exist. Convergence of this iterative method itself implies both existence and uniqueness of equilibrium asset prices.

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1. Introduction

One fundamental problem in economics is the pricing of an asset paying a stochastic cash flow with no natural termination point, such as a sequence of dividends.

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In discrete-time no-arbitrage environments, the equilibrium price process $\{P_t\}_{t \geqslant 0}$ associated with a dividend process $\{D_t\}_{t \geqslant 1}$ obeys

$$P_t = \mathbb{E}_t \, M_{t+1}(P_{t+1} + D_{t+1}) \quad \text{for all } t \geqslant 0, \tag{1}$$

where $\{M_t\}$ is the sequence of single period stochastic discount factors.[1] Two questions immediately arise in connection with these dynamics:

1. Given $\{D_t, M_t\}_{t \geqslant 1}$, does there exist a unique equilibrium price process?
2. How can we characterize and evaluate such prices whenever they exist?

These questions have become more pressing for two reasons. First, models of dividend processes and state price deflators are becoming more sophisticated, in an ongoing effort to better match financial data and resolve outstanding puzzles (see, e.g., recent iterations of the models in Campbell and Cochrane (1999), Barro (2006), or Bansal and Yaron (2004)). This complexity makes questions 1–2 challenging, especially in quantitative applications with discount rates close to the growth rates of underlying cash flows. There have been few sufficient conditions proposed that (a) imply existence and uniqueness of equilibrium prices, (b) are weak enough to be useful in modern quantitative analysis, and (c) are practical enough to implement in interesting applied settings.

The second reason that questions 1–2 have become more pressing is the accumulating evidence that nonlinearities embedded in the original models matter for quantitative analysis. For example, Pohl et al. (2018) and Lorenz et al. (2020) show that the log-linearization techniques used to solve asset pricing models can lead to large distortions in the equity premium and price volatility. These findings increase the need for practical methods for investigating the underlying structure of modern asset pricing models.

In this paper, we introduce a condition for existence and uniqueness of equilibria that is both weak enough to hold in realistic applications—in fact necessary as well as sufficient—and practical in the sense that testing the condition focuses on a single value. This value is referred to below as the *stability exponent*. To illustrate key ideas, consider the case of stationary dividend growth, which is the standard

[1] See, for example, Kreps (1981), Hansen and Richard (1987) or Duffie (2001). Here and below, prices are on ex-dividend contracts. (Cum-dividend contracts are a simple extension.)

assumption in quantitative applications. Seeking a stationary price-dividend ratio, we rewrite (1) as

$$\frac{P_t}{D_t} = \mathbb{E}_t \left[M_{t+1} \frac{D_{t+1}}{D_t} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right]. \tag{2}$$

Let $\Phi_{t+1} := M_{t+1} (D_{t+1}/D_t)$. For this class of models, the stability exponent is

$$\mathcal{L}_\Phi := \lim_{n \to \infty} \frac{\ln \psi_n}{n}, \tag{3}$$

where $\psi_n := \mathbb{E} \prod_{t=0}^{n-1} \Phi_{t+1}$ is the expectation of the n-period pricing kernel adjusted for dividend growth. Uncertainty in the discount process $\{M_t\}$ and dividend growth $\{D_{t+1}/D_t\}$ is driven by an irreducible state process, and $\mathbb{E}$ takes expectations over the unique stationary distribution.

We show that, in this setting, existence and uniqueness of an equilibrium price process is exactly equivalent to the statement $\mathcal{L}_\Phi < 0$. In addition, successive approximations converge globally to equilibrium prices if and only if $\mathcal{L}_\Phi < 0$. We also show that convergence of this algorithm itself implies that the limit is an equilibrium and, moreover, that $\mathcal{L}_\Phi < 0$. Therefore, convergence from a single initial condition implies that the limit is an equilibrium, and that these equilibrium prices are unique and globally attracting.

Interpreting the condition $\mathcal{L}_\Phi < 0$ is straightforward. Let $p_n(x) := \mathbb{E}_x \prod_{t=0}^{n-1} \Phi_{t+1}$ denote the price of a claim on the dividend paid out n periods ahead at the current state x, normalized by the current dividend. The pricing of these so-called dividend strips has been analyzed extensively, see, e.g., van Binsbergen et al. (2012) or van Binsbergen et al. (2013). Due to irreducibility, $\mathbb{E}_x$ can be replaced by $\mathbb{E}$ for limiting events, so $p_n(x) \approx \psi_n$ for large n. The condition then states that, asymptotically, prices of long-horizon dividend strips decay to zero at a geometric rate.[2]

The intuition is particularly simple in the case where dividends are stationary. We can then replace Φ_{t+1} in (3) with M_{t+1}. Now $p_n(x) := \mathbb{E}_x \prod_{t=0}^{n-1} M_{t+1}$ is the price of a risk-free zero-coupon bond with maturity n at current state x, so $y_n(x) := -\ln p_n(x)/n$ is the corresponding yield to maturity. Since $p_n(x) \approx \psi_n$ for large n, the condition $\mathcal{L}_\Phi < 0$ means that, in the limit, yields on risk-free long bonds are positive. This indicates a fundamental preference for current payoffs over future

[2] From (3) and $p_n(x) \approx \psi_n$ we have $p_n(x)^{1/n} \approx \exp(\mathcal{L}_\Phi)$ for large n, so the dividend strip price $p_n(x)$ goes to zero like $\exp(n\mathcal{L}_\Phi)$ when $\mathcal{L}_\Phi < 0$. The value $-\mathcal{L}_\Phi$ is the decay rate.

payoffs, which generates finite, well defined prices for stationary infinite horizon cash flows.

While $\mathcal{L}_\Phi < 0$ has a natural interpretation, it is striking that this condition is necessary as well as sufficient for existence, and hence exactly characterizes the set of models with well defined equilibrium prices. This result rests on irreducibility mentioned above, which is a mild condition, and a "local spectral radius" result due to Zabreiko et al. (1967) and Forster and Nagy (1991). Using this result, we show that, for any positive cash flow with finite first moment, the asymptotic mean growth rate of its discounted payoff stream is equal to the principal eigenvalue of an associated valuation operator, which is in turn equal to the exponential of $\mathcal{L}_\Phi$. If the principal eigenvalue equals or exceeds unity, then the sum of expected discounted payoffs grows without bound.

An operator-theoretic way to understand our results is to view Φ_{t+1} as a random "contraction factor" for an operator that has, as its fixed point, an equilibrium price function. If there is a constant θ with $\Phi_{t+1} \leqslant \theta < 1$ with probability one, then valuation equations (1) and (2) imply this operator will be a contraction of modulus θ, yielding existence of a unique equilibrium. However, in most applications, $\Phi_{t+1} > 1$ holds on a set of positive probability, due to the fact that payoffs in bad states have high value. Thus, a direct one step contraction argument is problematic. Hence we adopt the weaker condition $\mathcal{L}_\Phi < 0$, which requires instead that $\Phi_{t+1} < 1$ holds on average over the long run, and show that this is both necessary and sufficient.[3]

We also discuss methods for calculating $\mathcal{L}_\Phi$ when no analytical solution exists. Similar to Backus et al. (1989), we show that, when the state space is finite, the rate of decay of prices of long-term dividend strips and hence $\mathcal{L}_\Phi$ can be calculated using numerical linear algebra. For other cases, we propose a Monte Carlo method that involves simulating independent realizations of the pricing kernel.

[3]The growth rate $\mathcal{L}_\Phi$ is also connected to the integrated Lyapunov exponent (see, e.g., Knill (1992)), which, for the process $\{\Phi_t\}$, takes the form $\mathcal{I}_\Phi := \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^n \mathbb{E} \ln \Phi_t$. When $\{\Phi_t\}$ is stationary, this reduces to $\mathbb{E} \ln \Phi_t$. Jensen's inequality yields $\mathcal{I}_\Phi \leqslant \mathcal{L}_\Phi$, and, as $\mathcal{L}_\Phi < 0$ is necessary and sufficient for existence and uniqueness, the inequality $\mathcal{I}_\Phi < 0$ is necessary but not sufficient. This is because it takes into account only the marginal distribution of Φ_t, and hence ignores long epochs during which the SDF exceeds unity. Stability requires controlling persistence in the SDF process, which requires restrictions on the full joint distribution.

This method is inherently parallelizable, sufficiently accurate for the applications we consider, and well suited to settings where the state space is large.[4]

As one illustration of the method, we consider a model of asset prices with Epstein–Zin recursive utility, multivariate cash flows and time varying volatility studied in Schorfheide et al. (2018). Hitherto no results have been available on existence and uniqueness of equilibria in the underlying theoretical model. We show that $\mathcal{L}_\Phi < 0$ holds at and in the neighborhood of the benchmark parameterization in a global numerical approximation of the model. This indicates existence of a unique set of equilibrium prices, along with a globally convergent method of computing them. The fact that our conditions are necessary as well as sufficient allows us to examine how far this positive result can be pushed as we shift parameters relative to the benchmark.

We also encompass and extend the classic result of Lucas (1978), who studied a model with infinite state space and SDF of the form

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}. \tag{4}$$

Here $\{C_t\}$ is a stationary consumption process, $\beta \in (0, 1)$ is a state independent discount component and u is a period utility function. Using a change of variable, Lucas (1978) obtains a modified pricing operator with contraction modulus equal to β, and hence, by Banach's contraction mapping principle, a unique equilibrium price process. His theorem is a special case of our main result.

While Lucas (1978) frames his study in a space of bounded functions, our analysis admits unbounded solutions. This is achieved by embedding the equilibrium problem in a space of candidate solutions with finite first moments. Such a setting is arguably more natural for the study of forward looking stochastic sequences, since the forward looking restriction is itself stated in terms of expectations. Adopting this setting allows us to generalize the existence and uniqueness results for equilibrium prices obtained in Calin et al. (2005) and Brogueira and

[4] Appendix B provides details. Overall, we find that modern quantitative asset pricing studies are too complex—and too close to the boundary between stability and instability—to allow for successful use of purely analytical sufficient conditions.

Schütze (2017), which extend Lucas (1978) by allowing for habit formation and unbounded utility.[5]

Our work is also connected to the literature on stochastic discount factor decompositions found in Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Hansen (2012); Borovička et al. (2016); Christensen (2017); Qin and Linetsky (2017) and other recent studies. These decompositions are used to extract a permanent growth component and a martingale component from the stochastic discount process, with the rate in the permanent growth component being driven by the principal eigenvalue of the valuation operator associated with stochastic discount factor. We show that the log of this principal eigenvalue is equal to $\mathcal{L}_\Phi$ in our setting, using the local spectral radius result discussed above.

In addition, our work is related to Pohl et al. (2019) and Christensen (2020), who provide conditions for existence and uniqueness of recursive utilities in settings where the state space and rewards are unbounded. While the objects in play are different (recursive utilities vs asset prices), the techniques are related because both sets of problems treat forward looking recursions over unbounded state spaces driven by exogenous state processes. The connection can be summarized as follows: Our results are not applicable for most recursive utility problems concerning existence and uniqueness, where nonlinearities require specialized techniques (e.g., the Orlicz space methods in Christensen (2020) or Jensen-type bounds in Pohl et al. (2019)). At the same time, our methods have comparative advantage for asset pricing, because they exploit positivity and affine structure (which follow from nonexistence of arbitrage). This allows us to obtain conditions for existence and uniqueness that are necessary as well as sufficient.[6] In addition, we use this

[5]Not surprisingly, our results also generalize the simple risk neutral case $M_t \equiv \beta$, which is linear and hence easily treated by standard methods (see, e.g., Blanchard and Kahn (1980)). The existence of a unique equilibrium when $\beta \in (0, 1)$ is a special case of our results because the n period state price deflator is just β^n, so, by the definition of $\mathcal{L}_\Phi$ in (3) and with stationary consumption, we have $\mathcal{L}_\Phi = \ln \beta$. The condition $\beta \in (0, 1)$ therefore implies $\mathcal{L}_\Phi < 0$ and hence existence of a unique solution.

[6]By the same token, our work is connected to Borovička and Stachurski (2020), which also treats existence and uniqueness of recursive utilities, with an emphasis on Epstein–Zin preferences. As with Christensen (2020), the topics differ but the focus on forward looking recursions is shared. Regarding a comparison of content, most of the same comments apply. (Orlicz space methods in Christensen (2020) are replaced by monotone concave operator methods in Borovička

same no arbitrage structure, combined with properties of positive linear operators, to translate spectral radius conditions over valuation operators into the analytically and computationally convenient stability exponent $\mathcal{L}_\Phi$. Finally, we offer techniques for computing $\mathcal{L}_\Phi$ analytically, as well as via linear algebraic methods and through Monte Carlo.

The rest of the paper is structured as follows. The main results are presented in Section 2. Sections 3–5 treat applications with stationary dividend growth and Section 6 concludes. Appendix A discusses models with stationary dividends, rather than stationary dividend growth. A discussion of numerical methods for implementing our test can be found in Appendix B. Long proofs are deferred until Appendix C. Computer code that replicates our numerical results and figures can be found at `https://github.com/jstac/asset_pricing_code`.

2. A Necessary and Sufficient Condition

In this section we present our framework and state our main results.

2.1. **Environment.** We will work with the generic forward looking model

$$Y_t = \mathbb{E}_t \left[\Phi_{t+1}(Y_{t+1} + G_{t+1}) \right] \quad \text{for } t \geqslant 0, \tag{5}$$

where $\{(\Phi_t, G_t)\}$ is a given stochastic process, defined on some underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $\{Y_t\}$ is endogenous. While other interpretations are possible, it is convenient to refer to $\{\Phi_t\}$ as the *stochastic discount factor* and $\{G_t\}$ as the *cash flow*.

Equations (1) and (2) are special cases of this recursion. In the latter case, the price dividend ratio P_t/D_t is the endogenous process, $\Phi_{t+1} = M_{t+1}D_{t+1}/D_t$ is a growth-adjusted stochastic discount factor, and the cash flow is $G_{t+1} = 1$.

We say that a stochastic process $\{Y_t\}$ *solves* (5) if, with probability one, each Y_t is finite and (5) holds for all $t \geqslant 0$. To obtain a solution we require some auxiliary conditions on the state process, the cash flow and the stochastic discount process. The first is as follows:

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and Stachurski (2020).) However, Borovička and Stachurski (2020) is less comparable to the present paper than Christensen (2020), since, in the former, the focus is on compact states.

Assumption 2.1. For all $t \geqslant 0$, we have $\mathbb{P}\{\Phi_t > 0\} = 1$, while $G_t \geqslant 0$ and $G_t > 0$ with positive probability.

Positivity of Φ_t is equivalent to assuming no arbitrage (Hansen and Richard (1987), Lemma 2.3) and holds in all applications we consider. Provided that the focus is on nonnegative cash flows (e.g., dividends), the second condition is also innocuous, since $G_t = 0$ almost surely implies $Y_t = 0$ for all t.

To introduce the possibility of stationary Markov solutions, we assume that $\{\Phi_t\}$ and $\{G_t\}$ admit the representations

$$\Phi_{t+1} = \phi(X_t, X_{t+1}, \eta_{t+1}) \quad \text{and} \quad G_{t+1} = g(X_t, X_{t+1}, \eta_{t+1}) \tag{6}$$

where $\{X_t\}$ is an underlying X-valued state process, $\{\eta_t\}$ is a W-valued innovation sequence and ϕ and g are positive Borel measurable maps on $\mathsf{X} \times \mathsf{X} \times \mathsf{W}$. The sets X and W can be any separable and completely metrizable topological spaces. The representations in (6) replicate the multiplicative functional specifications in Hansen and Scheinkman (2009) and Hansen (2012).

The innovation process $\{\eta_t\}$ is assumed to be IID and independent of $\{X_t\}$, with common distribution ν. The state process is assumed to be stationary and Markovian with common marginal distribution π. The conditional distribution of X_{t+1} given $X_t = x$ is denoted by $\Pi(x, \mathrm{d}y)$. We use Π^n to represent n-step transition probabilities (see, e.g., Meyn and Tweedie (2009)).

Assumption 2.2. The state process $\{X_t\}$ is irreducible: for each Borel set $B \subset \mathsf{X}$ with $\pi(B) > 0$ and each $x \in \mathsf{X}$, there exists an $n \in \mathbb{N}$ such that $\Pi^n(x, B) > 0$.

Assumption 2.2 is a weak mixing condition on the state process that is satisfied in all applications we consider. For example, Mehra and Prescott (1985) use a discrete state space and require that the Markov chain is both irreducible and aperiodic. Similarly, in the long run risk model of Schorfheide et al. (2018), innovations have positive densities, which implies irreducibility.

A measurable function h from X to $\mathbb{R}$ is called a *Markov solution* to (5) if

$$h(X_t) = \mathbb{E}_t \left[\Phi_{t+1}(h(X_{t+1}) + G_{t+1})\right]$$

for all $t \geqslant 0$, which means that $\{Y_t\} := \{h(X_t)\}$ solves (5). Conditioning on $X_t = x$, we see that h will be a Markov solution if it is a fixed point of the *equilibrium price* *operator* T defined by

$$Th(x) = \mathbb{E}\left[\Phi_{t+1}\left(h(X_{t+1}) + G_{t+1}\right) \mid X_t = x\right]. \tag{7}$$

For each $p \geqslant 0$, we let $L_p(\mathsf{X}, \mathbb{R}, \pi)$ denote, as usual, the set of Borel measurable real-valued functions h defined on the state space X such that $\int |h(x)|^p \pi(\mathrm{d}x)$ is finite. Let $\mathcal{H}_p$ be all nonnegative functions in $L_p(\mathsf{X}, \mathbb{R}, \pi)$. This will be our candidate space, so if $p = 2$, say, then we seek solutions with finite second moment.[7]

Assumption 2.3. There exists a $p \geqslant 1$ such that $\mathbb{E}\left(\Phi_t G_t\right)^p < \infty$ and, in addition, the map $h \mapsto Vh$ defined by

$$Vh(x) = \mathbb{E}\left[\Phi_{t+1}\, h(X_{t+1}) \mid X_t = x\right] \tag{8}$$

is eventually compact as a linear operator from $L_p(\mathsf{X}, \mathbb{R}, \pi)$ to itself.

In what follows, we call V the *valuation operator*. The first part of Assumption 2.3, which requires that the cash flow process has p finite moments after discounting, is weakest when $p = 1$. In fact, this minimal restriction cannot be omitted, since (5) is not well defined without finiteness of first moments.[8] The "eventually compact" part of Assumption 2.3 is a regularity condition related to the notion of compact linear operators, stated formally in Appendix C. In Sections 3–5 we discuss how to test this condition and review its implications.[9]

2.2. **Existence and Uniqueness.** We introduce the *p-th order stability exponent* of the SDF process $\{\Phi_t\}$ as

$$\mathcal{L}_\Phi^p := \lim_{n \to \infty} \frac{1}{np} \ln \mathbb{E}\left\{\mathbb{E}_x \prod_{t=1}^n \Phi_t\right\}^p. \tag{9}$$

Here and below, $\mathbb{E}_x$ conditions on $X_0 = x$. The simplest case is when $p = 1$, since, by the Law of Iterated Expectations,

$$\mathcal{L}_\Phi := \mathcal{L}_\Phi^1 = \lim_{n \to \infty} \frac{1}{n} \ln\left\{\mathbb{E} \prod_{t=1}^n \Phi_t\right\}. \tag{10}$$

[7] In what follows, all notions of convergence refer to standard norm convergence in L_p. As usual, functions equal π-almost everywhere are identified. Appendix C gives more details.

[8] We might wish to choose p to be larger when possible, in order to impose more structure on our solution (e.g., finiteness of second moments for asymptotic results related to estimation).

[9] Analogous conditions can be found in the literature on eigenfunction decompositions of valuation operators. See, for example, Assumption 2.1 in Christensen (2017).

As discussed in the introduction, when $\{\Phi_t\}$ is the discount factor process, $-\mathcal{L}_\Phi$ can be interpreted as the yield on a zero-coupon bond with very long maturity. In the case of a growth-adjusted discount factor process from (2), $-\mathcal{L}_\Phi$ is the rate of decay of prices of long-maturity dividend strips.

Theorem 2.1. *If Assumptions 2.1–2.3 hold, then the limit in* (9) *exists and all of the following statements are equivalent:*

(a) $\mathcal{L}_\Phi^p < 0$.
(b) There exist h_0, h *in* $\mathcal{H}_p$ *such that* $T^n h_0 \to h$ *as* $n \to \infty$.
(c) A Markov solution h^* *exists in* $\mathcal{H}_p$.
(d) A unique Markov solution h^* *exists in* $\mathcal{H}_p$ *and* $T^n h \to h^*$ *for every* $h \in \mathcal{H}_p$.

If one and hence all of (a)–(d) are true, then h^* *satisfies*

$$h^*(x) = \sum_{n=1}^{\infty} \mathbb{E}_x \prod_{i=1}^{n} \Phi_i \, G_n \quad \text{for } \pi\text{-almost all } x \text{ in } \mathsf{X}. \tag{11}$$

The p in conditions (a)–(d) is from Assumption 2.3. Condition (b) is valuable from an applied perspective, since it shows that if iteration with T converges from some starting point, then the limit is necessarily a Markov solution, and, in fact is the only Markov solution in $\mathcal{H}_p$. Part (d) shows that successive approximations is globally convergent whenever $\mathcal{L}_\Phi^p < 0$.[10]

Remark 2.1. In Appendix C we use a local spectral radius result to show that, when Assumptions 2.1–2.3 hold, we have

$$\mathcal{L}_\Phi^p = \ln r(V) \tag{12}$$

[10]Successive approximations can be compared with other methods for solving asset pricing models, such as perturbations and projections (see, e.g., Pohl et al. (2018)). Successive approximations can be thought of either as a robust alternative or complementary in the following sense: While projection methods are fast, they almost always require tuning, as output can be sensitive to both the choice of basis functions and the solver used for the associated nonlinear equations. In such cases, the globally convergent successive approximations method can be employed to first compute the solution. Projection methods can then be tuned until they reliably reproduce it.

where $r(V)$ is the spectral radius of the operator V introduced in (8), when regarded as a linear self-map on $L_p(\mathsf{X}, \mathbb{R}, \pi)$. This result is central to the proof of Theorem 2.1 and used in later computations.[11]

Remark 2.2. The necessity component of Theorem 2.1 has strong implications. To understand it from an asset pricing perspective, we interpret g as defining a cash flow via (6) and view h satisfying $h = Th$ as a pricing function for this cash flow. Theorem 2.1 tells us that, when $\mathcal{L}_\Phi^p \geqslant 0$, *no* nontrivial cash flow with finite p-th moment can be priced under the discounting embedded in V. In other words, the only cash flow with finite price in $\mathcal{H}_p$ is the cash flow that pays zero with probability one.

It is also worth noting that, in view of (a) and (d) from Theorem 2.1, uniqueness of asset prices is never an issue under our assumptions. Either $\mathcal{L}_\Phi^p < 0$ and one solution exists, or $\mathcal{L}_\Phi^p \geqslant 0$ and no solutions exist.

3. Applications with Finite State Spaces

We now turn to applications of Theorem 2.1, beginning with the classic study of Mehra and Prescott (1985). Our objective in treating this model is to clarify the assumptions and results in Section 2 in a simple environment, before moving on to more complex applications. The following proposition will aid our analysis.

Proposition 3.1. *If Assumptions 2.1–2.2 hold and, in addition, the state space X is finite, then Assumption 2.3 also holds, for every $p \geqslant 1$, and $\mathcal{L}_\Phi^p = \mathcal{L}_\Phi$. In particular (b)–(d) of Theorem 2.1 all hold at every $p \geqslant 1$ if and only if $\mathcal{L}_\Phi < 0$.*

Proposition 3.1 applies whenever the state evolves as a finite, irreducible Markov chain—a common set up in quantitative applications.[12] The proposition has two key implications. One is that, in the finite state setting, we can always work with the simple exponent $\mathcal{L}_\Phi^1 = \mathcal{L}_\Phi$ from (10). The reason the stability exponent does

[11] The definition of the spectral radius is given in Appendix C. The spectral radius $r(V)$ also appears in the literature on stochastic discount factor decompositions discussed in the introduction, since, by the Krein–Rutman theorem, it equals the principal eigenvalue of the valuation operator V, which in turn determines the permanent growth component of the stochastic discount factor.

[12] See, for example, Backus et al. (1989), Weil (1989), Kocherlakota (1990), Alvarez and Jermann (2001), Cogley and Sargent (2008), Collin-Dufresne et al. (2016), or Martin and Ross (2019).

not depend on p is that moment conditions are irrelevant when X is finite, since all moments are finite for random variables supported on finite sets.

The second implication is that the eventual compactness condition in Assumption 2.3 is always satisfied when X is finite. Indeed, eventual compactness means that there exists a time horizon n such that the n-period valuation operator V^n is a compact linear operator (i.e., maps bounded sets of payoffs to relatively compact sets). This generalizes the idea that V^n has finite rank (i.e., maps into a finite dimensional range space). When X is finite, V is just a matrix and the range space of V^n is finite dimensional for all n, so eventual compactness certainly holds.

In Mehra and Prescott (1985), the price-dividend ratio $Q_t := P_t / D_t$ obeys (2) and the stochastic discount factor is given by (4). In other words,

$$Q_t = \mathbb{E}_t \left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{D_{t+1}}{D_t} (Q_{t+1} + 1) \right]. \tag{13}$$

Agents have CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}. \tag{14}$$

In equilibrium, $C_{t+1}/C_t = D_{t+1}/D_t = X_{t+1}$. The state space X is contained in $(0, \infty)$, so $X_t > 0$. Equation (13) becomes

$$Q_t = \beta \, \mathbb{E}_t \, X_{t+1}^{1-\gamma} \, [Q_{t+1} + 1], \tag{15}$$

which is a version of (5) with $\Phi_{t+1} = \beta X_{t+1}^{1-\gamma}$ and $G_{t+1} = 1$. Since elements of X are positive, Assumption 2.1 holds. Mehra and Prescott (1985) assume that $\Pi(x, y) := \mathbb{P}\{X_{t+1} = y \,|\, X_t = x\} > 0$ for all $x, y \in \mathsf{X}$, so the irreducibility condition in Assumption 2.2 holds. In view of Proposition 3.1, Assumption 2.3 is also valid and $\mathcal{L}_\Phi^p = \mathcal{L}_\Phi$ for all $p \geqslant 1$.

With Assumptions 2.1–2.3 all in force, Remark 2.1 applies, and we have $\mathcal{L}_\Phi = \ln r(V)$, where $r(V)$ is the spectral radius of the valuation operator

$$Vh(x) = \mathbb{E}_x \Phi_{t+1} \, h(X_{t+1}) = \beta \sum_{y \in \mathsf{X}} y^{1-\gamma} h(y) \Pi(x, y).$$

In the present setting, the operator V can be identified with the matrix $V(x, y) = \beta y^{1-\gamma} \Pi(x, y)$, and $r(V)$ is just the spectral radius of this matrix.

The intuition behind Remark 2.1 is straightforward for this model. Recall that $\mathcal{L}_\Phi$ is the long-term decay rate in prices. We now show that $\ln r(V)$ also has this interpretation. The largest eigenvalue of the valuation matrix, equal to the spectral radius $r(V)$, dominates long-term pricing. As the maturity n increases, prices of cash flows h behave as $V^n h \sim r(V)^n e$, where e is the eigenvector associated with the largest eigenvalue. The long-term decay rate in prices is, therefore,[13]

$$\lim_{n \to \infty} \frac{1}{n} \ln V^n h = \ln r(V). \tag{16}$$

Turning to Theorem 2.1, since Assumptions 2.1–2.3 all hold, statements (b)–(d) of Theorem 2.1 are valid if and only if $\mathcal{L}_\Phi = \ln r(V) < 0$. In Mehra and Prescott (1985), the transition probabilities and state space are given by

$$\Pi = \begin{pmatrix} \psi & 1-\psi \\ 1-\psi & \psi \end{pmatrix} \quad \text{and} \quad \mathsf{X} = \begin{pmatrix} 1+\mu+\delta \\ 1+\mu-\delta \end{pmatrix}$$

for parameters ψ, μ, δ. The baseline parameter values are $\psi = 0.43$, $\mu = 0.018$, $\delta = 0.036$, and $\beta = 0.99$. The authors experiment with different values of γ ranging from 1 to 10. With $\gamma = 2.5$, evaluating the spectral radius of V gives $\ln r(V) = -0.0348$, so the equivalent conditions (b)–(d) in Theorem 2.1 hold.

The fact that $\ln r(V) < 0$ implies existence of a unique Markov solution in this simple setting is well known: a Markov solution is a finite vector h satisfying $h = V(h + \mathbb{1})$, where $\mathbb{1}$ is the unit vector, and the Neumann Series Theorem yields a solution whenever $r(V) < 1$. Nonetheless, Theorem 2.1 is useful even in this very simple case. For example, it shows that $\ln r(V) < 0$ is not just sufficient but also necessary for existence of a finite price-dividend ratio.

Figure 1 helps illustrate the value of this exact delineation. The plot shows contour lines for the value of $\mathcal{L}_\Phi$ generated by the Mehra–Prescott model and a set of neighboring parameterizations. The horizontal and vertical axes show grid points for the parameters γ and δ respectively. The other parameters are held at the baseline parameterization. The contour line $\mathcal{L}_\Phi = 0$ is emphasized. Pairs (γ, δ) on the contour line $\mathcal{L}_\Phi = 0$, as well as those to the far left (low values of γ) and top right (high γ and δ) are where the stability condition fails. At other points, such as at the original parameterization used by Mehra and Prescott (1985), we have $\mathcal{L}_\Phi < 0$ and the stability condition holds.

[13] Relatedly, Backus et al. (1989) show that the yield on the long bond equals $-\ln r(V)$ for a valuation matrix V corresponding to $\Phi_{t+1} = \beta X_{t+1}^{-\gamma}$ in (15).

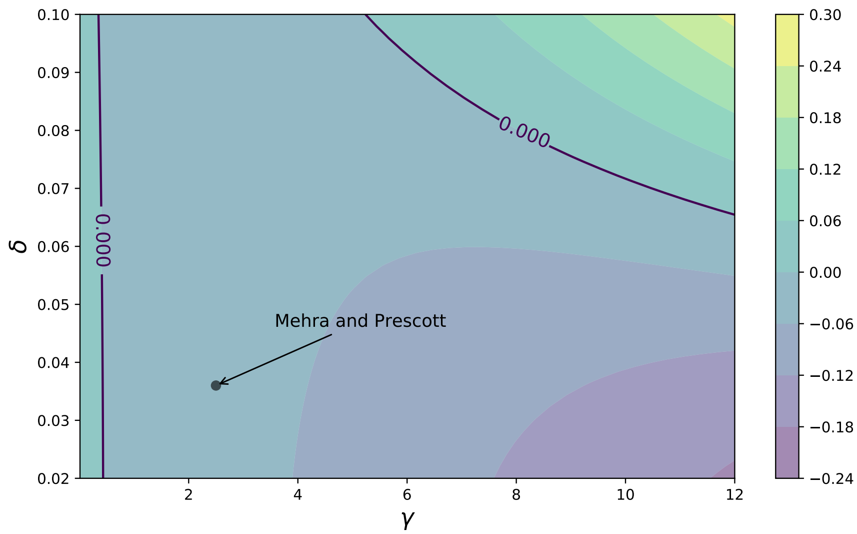

FIGURE 1. Contour plot of $\mathcal{L}_\Phi$ for the Mehra-Prescott model

Recall that $\mathcal{L}_\Phi$ can be interpreted as the (negative) rate of growth of the price of long-horizon consumptions strips as maturity increases. The nonmonotonicity of $\mathcal{L}_\Phi$ is a consequence of the changing strength of the substitution effect as the risk aversion parameter γ changes. An increase in the volatility parameter δ reduces the certainty equivalent value of future consumption, acting as a negative income effect. When $\gamma > 1$, the low willingness to substitute consumption across time dominates valuation. The value of future consumption in current consumption units, and hence $\mathcal{L}_\Phi$, increases as δ increases, and diverges to infinity in the moment when the condition $\mathcal{L}_\Phi < 0$ fails. On the other hand, when $\gamma < 1$, the income effect dominates, and $\mathcal{L}_\Phi$ becomes more negative as δ increases. A solution does not exist for low values of γ and δ because $\beta(1+\mu) > 1$. In the case of logarithmic utility ($\gamma = 1$), the income and substitution effect exactly offset each other, and $\mathcal{L}_\Phi = \log \beta < 0$.

4. APPLICATIONS WITH LOG LINEAR GROWTH

Next we study two asset pricing applications where dynamics are linear and shocks are Gaussian. The main objective of this section is to show how, in some settings, the value $\mathcal{L}_\Phi^p$ can be calculated analytically, as well as to derive intuition on the factors that determine $\mathcal{L}_\Phi^p$. We will also use the analytical solutions as a benchmark for testing numerical calculations (see Appendix B).

4.1. **CRRA Preferences and Log Linear Growth.** As a first step, we replace the finite state process in Mehra and Prescott (1985) with a linear Gaussian process. In particular, dividend and consumption growth are now assumed to obey the constant volatility specification from section I.A of Bansal and Yaron (2004), which is

$$\ln(D_{t+1}/D_t) = \mu_d + \varphi X_t + \sigma_d\, \xi_{t+1} \tag{17a}$$

$$\ln(C_{t+1}/C_t) = \mu_c + X_t + \sigma_c\, \epsilon_{t+1} \tag{17b}$$

$$X_{t+1} = \rho X_t + \sigma\, \eta_{t+1} \tag{17c}$$

Here $-1 < \rho < 1$ and $\{(\xi_t, \epsilon_t, \eta_t)\}$ is IID and standard normal in $\mathbb{R}^3$. This is a natural extension of the original model to a continuous state setting.

The model is otherwise unchanged. The SDF has the standard time separable form (4) and agents have CRRA utility. We solve for the price dividend ratio using (2), which means that, when connecting to the forward looking model (5), we take $G_t = 1$ and

$$\Phi_{t+1} = M_{t+1}\frac{D_{t+1}}{D_t} = \beta \exp\left\{(\mu_d + \varphi X_t + \sigma_d\, \xi_{t+1}) - \gamma(\mu_c + X_t + \sigma_c\, \epsilon_{t+1})\right\}. \tag{18}$$

(See the discussion following (5).)

Proposition 4.1. *For $\{\Phi_t\}$ in (18) and $p \geqslant 1$, we have*

$$\mathcal{L}_\Phi^p = \ln\beta + \mu_d - \gamma\mu_c + \frac{\sigma^2}{2}\frac{(\varphi-\gamma)^2}{(1-\rho)^2} + \frac{\sigma_d^2 + (\gamma\sigma_c)^2}{2}. \tag{19}$$

The proof of Proposition 4.1 starts from the definition in (9) and steps through straightforward calculations. The proof is in Appendix B.1.

The value of $\mathcal{L}_\Phi^p$ represents the long-run growth rate of the discounted dividend Φ_t. In expression (19), the term $\mu_d + \sigma_d^2/2 + [\varphi\sigma/(1-\rho)]^2/2$ corresponds to the long-run dividend growth rate, $\ln\beta - \gamma\mu_c + (\gamma\sigma_c)^2/2 + [\gamma\sigma/(1-\rho)]^2/2$ to the (negative of) the long-run discount rate, and $\varphi\gamma\sigma^2/(1-\rho)^2$ is the long-run covariance between the two.[14]

When do the conditions of Theorem 2.1 hold? Since $G_t = 1$ and Φ_t is given by (18), Assumption 2.1 is clearly valid. The state process (17c) is irreducible, so Assumption 2.2 holds. For the moment condition $\mathbb{E}(\Phi_t G_t)^p < \infty$ in Assumption 2.3, a finite p-th moment in (18) is required. This holds for all $p \geqslant 1$ in the current setting because the stationary distribution of X_t is Gaussian.

The only remaining issue is the eventual compactness of V in Assumption 2.3. In view of (20), the valuation operator V has the form

$$Vh(x) = \beta \exp\{ax + b\} \int h(y) q(x, y) \, \mathrm{d}y$$

for suitably chosen constants a and b, where q is the Gaussian transition density associated with (17c). From this expression it can be verified that Assumption 2.3 holds at $p = 2$ via Proposition C.1 in the appendix. This follows from the smoothing property of conditional expectations, which implies that the range of the operator V is not too irregular.

We conclude that, since Assumptions 2.1–2.3 are all satisfied, the conclusions of Theorem 2.1 hold if and only if the right hand side of (19) is negative.

4.2. **Habit Persistence.** There is a large literature on asset prices in the presence of consumption externalities and habit formation (see, e.g., Abel (1990) and Campbell and Cochrane (1999)). In this section we treat a relatively simple habit formation model and show how the stability exponent can be calculated analytically. In the process, we illustrate the value of Theorem 2.1 by substantially improving on the existence and uniqueness results in Calin et al. (2005).

In the "external" habit formation setting of Abel (1990) and Calin et al. (2005), the growth adjusted SDF takes the form

$$M_{t+1}\frac{D_{t+1}}{D_t} = k_0 \exp((1-\gamma)(\rho - \alpha)X_t) \tag{20}$$

[14] Notice that $\mathcal{L}^p_\Phi$ in (19) does not depend on p. In particular, we have $\mathcal{L}^p_\Phi = \mathcal{L}^1_\Phi := \mathcal{L}_\Phi$ for all p. This matches the finding that $\mathcal{L}^p_\Phi = \mathcal{L}_\Phi$ for all p in the finite dimensional case, as shown in Proposition 3.1. In other words, this constant volatility model is simple enough to retain key features of the finite dimensional setting.

where $k_0 := \beta \exp(b(1-\gamma) + \sigma^2(\gamma-1)^2/2)$ and α is a preference parameter. The state sequence $\{X_t\}$ obeys

$$X_{t+1} = \rho X_t + b + \sigma \eta_{t+1} \quad \text{with} \quad -1 < \rho < 1 \quad \text{and} \quad \{\eta_t\} \overset{\text{IID}}{\sim} N(0,1). \tag{21}$$

The parameter b is equal to $x_0 + \sigma^2(1-\gamma)$ where x_0 represents mean constant growth rate of the dividend of the asset.

The price-dividend ratio associated with this stochastic discount factor satisfies the forward recursion (2) and, by Theorem 2.1, there exists a unique price-dividend with finite second moment (we set $p = 2$ in Theorem 2.1) if $\mathcal{L}^2_\Phi < 0$ and Assumptions 2.1–2.3 are satisfied. Assumptions 2.1–2.3 can be verified when $p = 2$ in almost identical manner to the corresponding discussion in Section 4. Hence, by Theorem 2.1, a unique equilibrium price-dividend ratio with finite second moment exists if and only if $\mathcal{L}^2_\Phi < 0$.

An analytical expression for $\mathcal{L}^2_\Phi$ can be obtained using similar techniques to those employed in Section 4. Stepping through the algebra shows that

$$\mathcal{L}^2_\Phi = \ln k_0 + (1-\gamma)(\rho-\alpha)\frac{b}{1-\rho} + \frac{(1-\gamma)^2(\rho-\alpha)^2}{2}\frac{\sigma^2}{(1-\rho)^2}. \tag{22}$$

A unique equilibrium price-dividend ratio exists in $\mathcal{H}_2$ if and only this term is negative. The intuition behind the expression (22) is analogous to (19) in Section 4.

To give some basis for comparison, let us contrast the condition $\mathcal{L}^2_\Phi < 0$ with the sufficient condition for existence and uniqueness of an equilibrium price-dividend ratio found in Proposition 1 of Calin et al. (2005), which implies a one step contraction. Their test is of the form $\tau < 1$, where τ depends on the parameters of the model (see Equation (7) of Calin et al. (2005) for details). Since the condition $\mathcal{L}^2_\Phi < 0$ requires only eventual contraction, rather than one step contraction, we can expect it to be significantly weaker than the condition of Calin et al. (2005).

Figure 2 supports this conjecture. The left sub-figure shows $\ln \tau$ at a range of parameterizations. The right sub-figure shows $\mathcal{L}^2_\Phi$ at the same parameters, evaluated using (22). The horizontal and vertical axes show grid points for the parameters β and σ respectively. For both sub-figures, (β, σ) pairs with test values strictly less than zero (points to the south west of the 0.0 contour line) are where the respective condition holds. Points to the north west of this contour line are where it fails.[15]

[15] The parameters held fixed in Figure 2 are $\rho = -0.14$, $\gamma = 2.5$, $x_0 = 0.05$ and $\alpha = 1$.

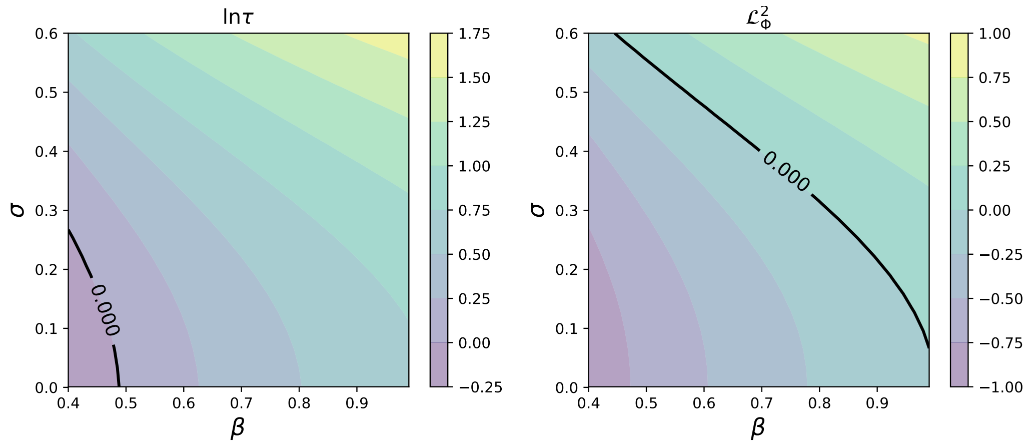

FIGURE 2. Alternative tests of stability for the habit formation model

Inspection of the figure shows that the sufficient condition in Calin et al. (2005) fails for some empirically relevant parameterizations that have unique stationary Markov equilibria. Note also that, because $\mathcal{L}_\Phi^2 < 0$ is both necessary and sufficient in our setting, the 0.0 contour line in the right sub-figure is an exact delineation between stable and unstable parameterizations.

5. APPLICATIONS WITH LONG-RUN RISK

In this section we continue to apply Theorem 2.1, but now in the presence of more sophisticated models, with time-varying risk. Our aim is to show how Theorem 2.1 can clearly delineate between well-defined models and models with no solution, even in settings with features such as recursive preferences and nonlinear dynamics (e.g., stochastic volatility). We focus on the class of long-run risk models first developed by Bansal and Yaron (2004), which have generated insights in a range of quantitative applications.

5.1. **Long-Run Risk With Stochastic Volatility.** Next we turn to an asset pricing model with Epstein–Zin utility and stochastic volatility in cash flow and consumption estimated by Bansal and Yaron (2004). Preferences are represented by the continuation value recursion

$$V_t = \left[(1-\beta) C_t^{1-1/\psi} + \beta \left\{ \mathcal{R}_t \left(V_{t+1} \right) \right\}^{1-1/\psi} \right]^{1/(1-1/\psi)}, \tag{23}$$

where $\{C_t\}$ is the consumption path and $\mathcal{R}_t$ is the certainty equivalent operator

$$\mathcal{R}_t(Y) := (\mathbb{E}_t Y^{1-\gamma})^{1/(1-\gamma)}. \tag{24}$$

The parameter $\beta \in (0,1)$ is a time discount factor, γ governs risk aversion and ψ is the intertemporal elasticity of substitution. Dividends and consumption grow according to

$$\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_t \, \eta_{c,t+1}, \tag{25a}$$

$$\ln(D_{t+1}/D_t) = \mu_d + \alpha z_t + \varphi_d \, \sigma_t \, \eta_{d,t+1}, \tag{25b}$$

$$z_{t+1} = \rho z_t + \varphi_z \, \sigma_t \, \eta_{z,t+1}, \tag{25c}$$

$$\sigma_{t+1}^2 = \max\left\{ v \, \sigma_t^2 + d + \varphi_\sigma \, \eta_{\sigma,t+1}, \, 0 \right\}. \tag{25d}$$

Here $\{\eta_{i,t}\}$ are IID and standard normal for $i \in \{d, c, z, \sigma\}$. The state X_t can be represented as $X_t = (z_t, \sigma_t)$. The (growth adjusted) SDF process associated with this model is

$$\Phi_{t+1} := M_{t+1} \frac{D_{t+1}}{D_t} = \beta^\theta \frac{D_{t+1}}{D_t} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{W_{t+1}}{W_t - 1} \right)^{\theta - 1}, \tag{26}$$

where W_t is the aggregate wealth-consumption ratio and $\theta := (1-\gamma)/(1-1/\psi)$.[16] To obtain the aggregate wealth-consumption ratio $\{W_t\}$ we exploit the fact that $W_t = w(X_t)$ where w solves the Euler equation

$$\beta^\theta \, \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{w(X_{t+1})}{w(X_t) - 1} \right)^{\theta} \right] = 1.$$

Rearranging and using the expression for consumption growth given above, this equality can be expressed as

$$w(z, \sigma) = 1 + [K w^\theta(z, \sigma)]^{1/\theta},$$

where K is the operator

$$K g(z, \sigma) = \beta^\theta \exp\left\{ (1-\gamma)(\mu_c + z) + \frac{(1-\gamma)^2 \sigma^2}{2} \right\} \Pi g(z, \sigma) \tag{27}$$

In this expression, $\Pi g(z, \sigma)$ is the expectation of $g(z_{t+1}, \sigma_{t+1})$ given the state's law of motion, conditional on $(z_t, \sigma_t) = (z, \sigma)$.

The existence of a unique solution $w = w^*$ to (5.1) in $\mathcal{H}_1$ under the parameterization used in Bansal and Yaron (2004) is established in Borovička and Stachurski

[16] For a derivation see, for example, Bansal and Yaron (2004), p. 1503.

(2020) when the innovation terms $\{\eta_{i,t}\}$ are truncated, so that the state space is a compact subset of $\mathbb{R}^2$. In what follows, we compute w^* using the iterative method described in Borovička and Stachurski (2020) and recover W_t as $w^*(X_t)$ for each t.

As discussed in detail in Appendix B, to approximate the stability exponent $\mathcal{L}_\Phi$, we can use Monte Carlo, generating independent paths for the SDF process $\{\Phi_t\}$ and averaging over them to estimate the expectation on the right hand side of (10). In computing the product $\prod_{t=1}^n \Phi_t$ we used (25) and (26) to express it as

$$\prod_{t=1}^{n} \Phi_t = (\beta^\theta \exp(\mu_d - \gamma\mu_c))^n \\ \times \exp\left((\alpha - \gamma) \sum_{t=1}^{n} z_t - \gamma \sum_{t=1}^{n} \sigma_t \eta_{c,t+1} + \varphi_d \sum_{t=1}^{n} \sigma_t \eta_{d,t+1} + (\theta - 1) \sum_{t=1}^{n} \hat{w}_t \right), \quad (28)$$

where $\hat{w}_{t+1} = \ln[W_{t+1}/(W_t - 1)]$.

At the parameter values using in Bansal and Yaron (2004) and based on the Monte Carlo method discussed above, we estimate that $\mathcal{L}_\Phi = -0.00388$ implying the existence of a unique equilibrium price-dividend ratio function in $\mathcal{H}_1$.[17] While this value is close to zero, we find that significant shifts in parameters are required to cross the contour $\mathcal{L}_\Phi = 0$.

For example, Figure 3 shows $\mathcal{L}_\Phi$ calculated at a range of parameter values in the neighborhood of the Bansal and Yaron (2004) specification via a contour map. The parameter α is varied on the horizontal axis, while μ_d is on the vertical axis. Other parameters are held fixed at the Bansal and Yaron (2004) values. The black contour line shows the boundary between stability and instability. Not surprisingly, the test value increases with the cash flow growth rate μ_d. In this region of the parameter space, it also declines with α, because an increase in α with $\gamma > \alpha$ reduces the covariance between cash flow growth and discounting captured by the term $(\alpha - \gamma)\sum_{t=1}^n z_t$ in (28).

[17]The reported value is the mean of 1,000 draws of $\mathcal{L}_\Phi(n, m)$, where the latter is defined in (34) of Appendix B. For each draw, n and m in in this calculation were set to 1,000 and 10,000 respectively. The standard error for the mean was approximately 0.0001. Following Bansal and Yaron (2004), the parameters used were $\gamma = 10.0$, $\beta = 0.998$, $\psi = 1.5$ $\mu_c = 0.0015$, $\rho = 0.979$, $\varphi_z = 0.044$, $v = 0.987$, d =7.9092e-7, φ_σ =2.3e-6. $\mu_d = 0.0015$, $\alpha = 3.0$ and $\varphi_d = 4.5$. See table IV on page 1489.

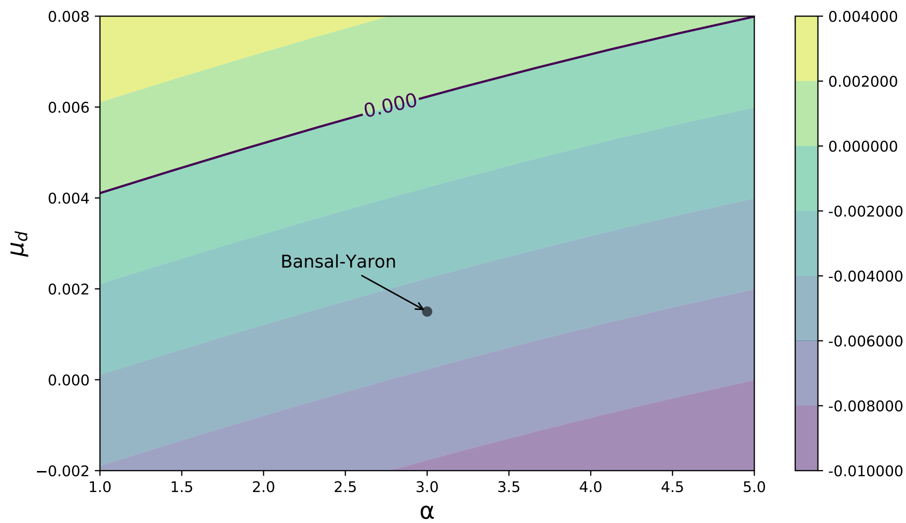

FIGURE 3. The exponent $\mathcal{L}_\Phi$ for the Bansal–Yaron model

5.2. **Long-Run Risk Part II.** Now we repeat the analysis in Section 5.1 but using instead the dynamics for consumption and dividends in Schorfheide et al. (2018), which are given by

$$
\begin{aligned}
\ln(C_{t+1}/C_t) &= \mu_c + z_t + \sigma_{c,t}\,\eta_{c,t+1},\\
\ln(D_{t+1}/D_t) &= \mu_d + \alpha z_t + \delta\sigma_{c,t}\,\eta_{c,t+1} + \sigma_{d,t}\,\eta_{d,t+1},\\
z_{t+1} &= \rho\, z_t + (1-\rho^2)^{1/2}\,\sigma_{z,t}\,v_{t+1},\\
\sigma_{i,t} &= \phi_i\,\bar{\sigma}\exp(h_{i,t}),\\
h_{i,t+1} &= \rho_{h_i} h_i + \sigma_{h_i}\xi_{i,t+1}, \quad i \in \{z, c, d\}.
\end{aligned}
$$

The innovation vectors $\eta_t = (\eta_{c,t}, \eta_{d,t})$ and $\xi_t := (v_t, \xi_{z,t}, \xi_{c,t}, \xi_{d,t})$ are IID over time, mutually independent and standard normal in $\mathbb{R}^2$ and $\mathbb{R}^4$ respectively. The state can be represented as $X_t := (z_t, h_{z,t}, h_{c,t}, h_{d,t})$. Otherwise the analysis and methodology radius is similar to Section 5.1. The product of the growth adjusted stochastic discount factors over n period from $t = 1$ is

$$\prod_{t=1}^{n} \Phi_t = (\beta^\theta \exp(\mu_d - \gamma\mu_c))^n$$
$$\exp\left((\alpha - \gamma) \sum_{t=1}^{n} z_t + (\delta - \gamma) \sum_{t=1}^{n} \sigma_{c,t}\eta_{c,t+1} + \sum_{t=1}^{n} \sigma_{d,t}\eta_{d,t+1} + (\theta - 1) \sum_{t=1}^{n} \hat{w}_t \right)$$

As in Section 5.1, we generate this product many times and then average to obtain an approximation of $\mathcal{L}_\Phi$. At the parameterization used in Schorfheide et al. (2018), this evaluates to -0.001, indicating the existence of a unique equilibrium price dividend ratio.[18]

Figure 4 shows the stability exponent $\mathcal{L}_\Phi$ calculated at a range of parameter values in the neighborhood of the Schorfheide et al. (2018) specification. The parameter ϕ_d is varied on the horizontal axis, while μ_d is on the vertical axis. Other parameters are held fixed at the Schorfheide et al. (2018) values. The interpretation is analogous to that of Figure 3 from Section 5.1, as is the method of computation, with the dark contour line shows the exact boundary between stability and instability. Increases in both μ_d and ϕ_d increase the long-run growth rate of the level of the discounted cash flow, and hence increase $\mathcal{L}_\Phi$. As with Figure 3, significant deviations in estimated parameter values are required to change the sign of $\mathcal{L}_\Phi$.

6. Conclusion

In this paper we developed a practical test for existence and uniqueness of equilibrium asset prices in infinite horizon arbitrage free settings. By seeking restrictions that ensure contraction occurs "on average, eventually," we obtained a test that is necessary as well as sufficient, and hence yields an exact delineation between stable and unstable models. Computational techniques are provided to ensure that the test can be implemented in realistic quantitative applications.

[18]We used the posterior mean values from Schorfheide et al. (2018), setting $\beta = 0.999$, $\gamma = 8.89$, $\psi = 1.97$, $\mu_c = 0.0016$, $\rho = 0.987$, $\phi_z = 0.215$, $\bar{\sigma} = 0.0032$, $\phi_c = 1.0$, $\rho_{hz} = 0.992$, $\sigma_{hz} = \sqrt{0.0039}$, $\rho_{hc} = 0.991$, $\sigma_{hc} = \sqrt{0.0096}$, $\mu_d = 0.001$, $\alpha = 3.65$, $\delta = 1.47$, $\phi_d = 4.54$, $\rho_{hd} = 0.969$, and $\sigma_{hd} = \sqrt{0.0447}$. We set $n = 1,000$ and $m = 10,000$, and then drew 1,000 observations of the statistic $\mathcal{L}_\Phi(n, m)$, as defined in (34) of Appendix B. The mean of these 1,000 draws was -0.00103, with standard error 0.000008.

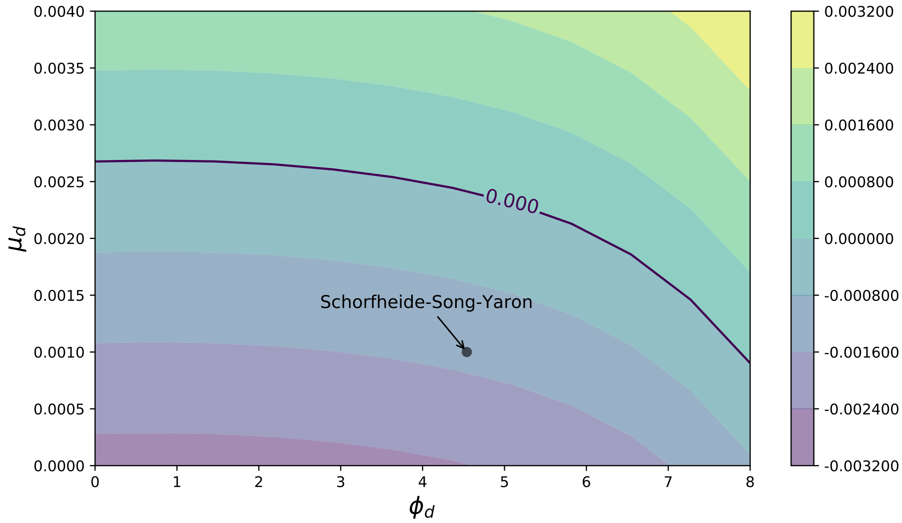

FIGURE 4. The exponent $\mathcal{L}_\Phi$ for the Schorfheide–Song–Yaron model

It is natural to ask whether or not our results extend to a continuous time setting. We have provided an online appendix which shows that, at least in simple cases, the answer is affirmative. However, we treat no substantial applications in that note and only briefly touch on interesting connections between infinitesimal descriptions and stability results. We hope that at least some readers will pursue this research further.

Although we focused on consumption-based asset pricing models, the theoretical results apply in the same way to other no-arbitrage settings where asset prices can be represented using recursion (1) with a positive marginal rate of substitution. Embedding this analysis into frameworks with endogenously determined consumption is left to future research.

APPENDIX A. MODELS WITH STATIONARY DIVIDENDS

In this section we discuss asset pricing with stationary dividends, rather than stationary dividend growth (which is a more standard assumption in quantitative analysis). This topic is mainly of theoretical and historical interest. One aim is to extend the classical result on existence and uniqueness of equilibrium asset prices obtained using contraction mapping arguments in Lucas (1978).

In that study, the price process obeys (1), where P_t is the price of a claim to the aggregate endowment stream $\{D_t\}$, and the stochastic discount factor is as given in (4). In equilibrium, C_t is equal to an endowment D_t, which is a positive and continuous function of a stationary Markov process $\{X_t\}$. Following Lucas (1978) we divide the fundamental asset pricing equation (1) through by $u'(C_t)$ and set $D_t = C_t$ for all t, obtaining

$$Y_t = \beta \, \mathbb{E}_t[Y_{t+1} + u'(C_{t+1})C_{t+1}] \tag{29}$$

where $Y_t := P_t \, u'(C_t)$. This is a version of (5) with $\Phi_t = \beta$ and $G_t = u'(C_t)C_t$.

Lucas (1978) requires that the utility function u is bounded, in order to employ a contraction mapping theorem in a space of bounded functions. This assumption is violated in almost all quantitative applications. To address this issue, Brogueira and Schütze (2017) take $\mathsf{X} = \mathbb{R}$, set utility to be CRRA as in (14), and suppose that $C_t = D_t = c(X_t)$ where $c(x) := a \exp(x)$ for some $a > 0$. For the state process $\{X_t\}$ they suppose that $\{X_t\}$ has a Gaussian density kernel $q(x, y)$ of the form $q(x, y) = N(\rho x, \sigma)$ for some $\sigma > 0$ and $|\rho| < 1$. The constant discount parameter β is assumed to satisfy $\ln \beta < -(1-\gamma)^2 \sigma^2 / 2$.

The conditions of Theorem 2.1 hold under these conditions when $p = 2$. Assumptions 2.1 and 2.2 are obviously true. The moment condition in Assumption 2.3 holds when $p = 2$ because

$$(\Phi_t G_t)^2 = [\beta u'(c(X_t))c(X_t)]^2 = \beta^2 \exp(2(1-\gamma)X_t).$$

The expectation of this term is finite because X_t is Gaussian. In addition, $Vh(x) = \beta \int h(y) q(x, y) \, \mathrm{d}y$ is an eventually compact linear operator on $L_2(\mathsf{X}, \mathbb{R}, \pi)$, as shown in Proposition C.1. Finally, since $\Phi_t = \beta$ for all t, we have $\mathcal{L}_\Phi^2 = \ln \beta < 0$. Hence, the conclusions of Theorem 2.1 all hold. In particular, a unique equilibrium price process with finite second moment exists. Notice that we did not require the stronger restriction $\ln \beta < -(1-\gamma)^2 \sigma^2 / 2$ from Brogueira and Schütze (2017).

Appendix B. Computing the Stability Exponent

The stability exponent $\mathcal{L}_\Phi^p$ plays a key role our results. In some cases it can be calculated analytically, as in (19) or (22). In others it needs to be computed. We begin with a discussion of the first case.

B.1. **Analytical Results.** In this section we provide the proof of Proposition 4.1, which illustrates how $\mathcal{L}_\Phi^p$ can be calculated analytically in a constant volatility setting.

Proof of Proposition 4.1. From (18), we have

$$\prod_{i=1}^n \Phi_i = \beta^n \exp\left\{ n(\mu_d - \gamma\mu_c) + (\varphi - \gamma)\sum_{i=1}^n X_i + \sigma_d \sum_{i=1}^n \xi_i - \gamma\sigma_c \sum_{i=1}^n \epsilon_i \right\}.$$

Using (17c), we then have

$$\left(\mathbb{E}_x \prod_{i=1}^n \Phi_i\right)^p = \beta^{np} \exp(pa_n x + pb_n), \tag{30}$$

where $a_n := (\varphi - \gamma)\rho(1-\rho^n)/(1-\rho)$ and

$$b_n := n(\mu_d - \gamma\mu_c) + \frac{(\varphi-\gamma)^2 s_n^2 + n\sigma_d^2 + n(\gamma\sigma_c)^2}{2}.$$

Here s_n^2 is the variance of $\sum_{i=1}^n X_i$. The next step in calculating $\mathcal{L}_\Phi^p$ is to take the unconditional expectation of (30), which amounts to integrating with respect to the stationary distribution $\pi = N(0, \sigma^2/(1-\rho^2))$. This yields

$$\mathbb{E}\left(\mathbb{E}_x \prod_{i=1}^n \Phi_i\right)^p = \beta^{np} \exp\left(\frac{(pa_n\sigma)^2}{2(1-\rho^2)} + pb_n\right),$$

and hence

$$\mathcal{L}_\Phi^p = \lim_{n\to\infty}\left\{\ln\beta + \frac{p}{n}\frac{(a_n\sigma)^2}{2(1-\rho^2)} + \frac{b_n}{n}\right\} = \ln\beta + \lim_{n\to\infty}\frac{b_n}{n}, \tag{31}$$

where the second equality uses the fact that a_n converges to a finite constant. Some algebra yields

$$\frac{s_n^2}{n} = \frac{\sigma^2}{1-\rho^2}\left\{1 + \frac{2(n-1)}{n}\frac{\rho}{1-\rho} - \frac{2\rho^2}{n}\cdot\frac{1-\rho^{n-1}}{(1-\rho)^2}\right\}. \tag{32}$$

Combining this with (31), we find that (19) holds. □

B.2. **Discretization.** If the state space is finite, then, as discussed in Section 3, the exponent $\mathcal{L}_\Phi^p = \mathcal{L}_\Phi$ is equal to the log of the spectral radius of a valuation matrix and can therefore be obtained by numerical linear algebra. This leads to the following idea for handling settings where the state space is infinite and no analytical expression for $\mathcal{L}_\Phi$ exists: discretize the state process and then proceed as for the finite state case. In this section we investigate whether or not this procedure leads to a good approximation to the value of $\mathcal{L}_\Phi^p$ from the original (infinite state) model.

To answer this question, we will use the model investigated just above, in Appendix B.1. This is convenient because, as shown in that section, an analytical expression for $\mathcal{L}_\Phi^p$ exists. Existence of an analytical expression allows us to make a careful comparison between the true solution and the approximation produced by discretization.

Our first step is to discretize the Gaussian AR(1) state process (17c) using the method of Rouwenhorst (1995). This produces a finite Markov matrix Π and finite state space with typical elements x, y. In view of (18), the valuation matrix V corresponding to this discretized model is given by

$$V(x,y) := \beta \exp\left[\mu_d - \gamma\mu_c + (1-\gamma)x + \frac{\sigma_d^2 + (\gamma\sigma_c)^2}{2}\right]\Pi(x,y). \tag{33}$$

We calculate the spectral radius $r(V)$ using linear algebra routines and, from there, compute the associated value for the stability exponent via (12). Finally, we compare the result with the true value of $\mathcal{L}_\Phi$ obtained from the analytical expression (19).

Figure 5 shows this comparison when the utility parameter γ is set to 2.5 and the consumption and dividend parameters are set to the values in table I of Bansal and Yaron (2004).[19] The vertical axis shows the value of $\mathcal{L}_\Phi$. The horizontal axis shows the level of discretization, indexed by the number of states for $\{X_t\}$ generated at the Rouwenhorst step. The true value of $\mathcal{L}_\Phi$ at these parameters, as calculate from (19), is -0.0031545. The discrete approximation of $\mathcal{L}_\Phi$ is accurate up to six decimal places whenever the state space has more than 6 elements. Thus, the discrete approximation is sufficiently accurate to implement the test $\mathcal{L}_\Phi < 0$ even for relatively coarse discretizations. Moreover, as shown in the figure, the approximation of $\mathcal{L}_\Phi$ converges to the true value as the number of states increases. We experimented with other parameter values and found similar results.

B.3. **A Monte Carlo Method.** Discretization works well for low dimensional state processes but is susceptible to the curse of dimensionality. For this reason, we also propose a Monte Carlo method that requires only the ability to simulate the SDF

[19] In particular, $\mu_c = \mu_d = 0.0015$, $\rho = 0.979$, $\sigma = 0.00034$, $\sigma_c = 0.0078$, $\sigma_d = 0.035$ and $\varphi = 1.0$.

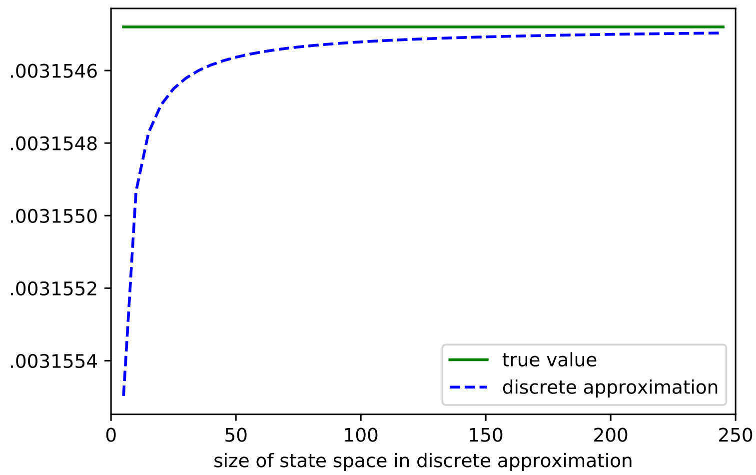

FIGURE 5. Accuracy of discrete approximation of $\mathcal{L}_\Phi$

process $\{\Phi_t\}$. As well as being less susceptible to the curse of dimensionality, this method has the advantage that simulation of the SDF process can be targeted for parallelization across CPUs or GPUs.

The idea behind the Monte Carlo method is to approximate $\mathcal{L}_\Phi$ via

$$\mathcal{L}_\Phi(n, m) := \frac{1}{n} \ln \left\{ \frac{1}{m} \sum_{j=1}^{m} \prod_{i=1}^{n} \Phi_i^{(j)} \right\}, \tag{34}$$

where each $\Phi_1^{(j)}, \ldots, \Phi_n^{(j)}$ is an independently simulated path of $\{\Phi_t\}$, and n and m are suitably chosen integers. The idea relies on the strong law of large numbers, which yields $\frac{1}{m} \sum_{j=1}^{m} \prod_{i=1}^{n} \Phi_i^{(j)} \to \mathbb{E} \prod_{i=1}^{n} \Phi_i$ with probability one, combined with the fact that $Z_n \to Z$ almost surely implies $g(Z_n) \to g(Z)$ almost surely whenever $g \colon \mathbb{R} \to \mathbb{R}$ is continuous.

These are asymptotic results. Table 1 tests finite sample behavior. We again use the constant volatility model from Section 4, comparing Monte Carlo approximations of $\mathcal{L}_\Phi$ with the true value obtained from in (19). Consumption and dividend growth parameters are as in footnote 19. The true value of $\mathcal{L}_\Phi$ is -0.0031545, as shown in the caption for the table. The interpretation of n and m in the table is consistent with the left hand side of (34). For each n, m pair, we compute $\mathcal{L}_\Phi(n, m)$ 1,000 times using independent draws and present the mean and their standard

TABLE 1. Monte Carlo spectral radius estimates when $\mathcal{L}_\Phi = -0.0031545$

	m = 1000	m = 2000	m = 3000	m = 4000	m = 5000
n = 250	-0.0033183	-0.0032524	-0.0032434	-0.0032533	-0.0032353
	(0.000003)	(0.000002)	(0.000001)	(0.000001)	(0.000001)
n = 500	-0.0032045	-0.0032149	-0.0031948	-0.0031907	-0.0031922
	(0.000002)	(0.000001)	(0.000001)	(0.000001)	(0.000001)
n = 750	-0.0031985	-0.0031841	-0.0031748	-0.0031784	-0.0031890
	(0.000002)	(0.000001)	(0.000001)	(0.000001)	(0.000001)

error in the corresponding cell. We find that the Monte Carlo approximation is accurate up to four decimal places when $n = 750$ and standard deviations are small. At least for this model, the Monte Carlo method can determine the sign of $\mathcal{L}_\Phi$.

APPENDIX C. PROOFS

If $\mathcal{E}$ is a Banach lattice, then an *ideal* in $\mathcal{E}$ is a vector subspace L of $\mathcal{E}$ with $x \in L$ whenever $|x| \leqslant |y|$ and $y \in L$. The *spectral radius* of a bounded linear operator M from $\mathcal{E}$ to itself is the supremum of $|\lambda|$ for all λ in the spectrum of A. The operator M is called *compact* if the image under M of the unit ball in $\mathcal{E}$ has compact closure. M is called *eventually compact* if there exists an $i \in \mathbb{N}$ such that M^i is compact. M is called *positive* if it maps the positive cone of $\mathcal{E}$ into itself. A positive linear operator M is called *irreducible* if the only closed ideals $J \subset \mathcal{E}$ satisfying $M(J) \subset J$ are $\{0\}$ and $\mathcal{E}$. See Abramovich et al. (2002) or Meyer-Nieberg (2012) for more details.

If X is an Polish space, π is a finite Borel measure on X and $p \geqslant 1$, then $L_p(\pi) := L_1(\mathsf{X}, \mathbb{R}, \pi)$ denotes is the set of all Borel measurable functions f from X to $\mathbb{R}$ satisfying $\int |f|^p \, \mathrm{d}\pi < \infty$. The norm on $L_p(\pi)$ is $\|f\| := (\int |f|^p \, \mathrm{d}\pi)^{1/p}$. Functions equal π-almost everywhere are identified. Convergence on $L_p(\pi)$ is with respect to the norm topology generated by $\| \cdot \|$. We write $f \leqslant g$ if $f \leqslant g$ pointwise π-almost everywhere, and $f \ll g$ if $f < g$ holds pointwise π-almost everywhere. The positive cone of $L_p(\pi)$ is all $f \in L_p(\pi)$ with $f \geqslant 0$. We denote this set by $\mathcal{H}_p$, which conforms with our previous definition (cf., Theorem 2.1).

C.1. **Operator Compactness in Spaces of Summable Functions.** Assumption 2.3 requires that V is eventually compact as a linear map from $L_p(\mathsf{X}, \mathbb{R}, \pi)$ to itself. Here we give a sufficient condition focused on the applications in Section 3. Take $\mathsf{X} = \mathbb{R}$ and $p = 2$. In the proposition below, q is a stochastic density kernel on $\mathbb{R}^2$ with stationary density π and two step density kernel q^2.

Proposition C.1. *Let M be an operator that maps f in $L_2(\pi)$ into*

$$Mf(x) = g(x) \int f(y) q(x,y) \, \mathrm{d}y \qquad (x \in \mathbb{R}), \tag{35}$$

where g is a measurable function from $\mathbb{R}$ to $\mathbb{R}_+$. If q is time-reversible and

$$\int g(x) q^2(x,x) \, \mathrm{d}x < \infty, \tag{36}$$

then M is a compact linear operator on $L_2(\pi)$.[20]

Proof. We can express the operator M as

$$Mf(x) = \int f(y) k(x,y) \pi(y) \, \mathrm{d}y \quad \text{where} \quad k(x,y) := \frac{g(x) q(x,y)}{\pi(y)}.$$

By theorem 6.11 of Weidmann (2012), the operator M will be Hilbert–Schmidt in $L_2(\pi)$, and hence compact, if the kernel k satisfies

$$\int \int k(x,y)^2 \pi(x) \pi(y) \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

Using the definition of k and the time-reversibility of q, this translates to

$$\int g(x) \int q(x,y) q(y,x) \, \mathrm{d}y \, \mathrm{d}x < \infty.$$

This completes the proof because, by definition, $q^2(x,x) = \int q(x,y) q(y,x) \, \mathrm{d}y$. $\square$

C.2. **Remaining Proofs.** Throughout the following we impose Assumptions 2.1–2.3. The symbol p represents the constant in Assumption 2.3. As before, Π is a stochastic kernel on X and $\{X_t\}$ is a stationary Markov process on X with stochastic kernel Π and common marginal distribution π.[21] The symbol $\mathbb{E}_x$ will indicate

[20] The statement that q is *time-reversible* means that $q(x,y)\pi(x) = q(y,x)\pi(y)$ for all $x, y \in \mathbb{R}$. A number of our results use the fact that $q(x, \cdot) = N(\rho x, \sigma^2)$ for some $\sigma > 0$ and $|\rho| < 1$ implies that q is time-reversible. See, e.g., O'Donnell (2014).

[21] In other words, Π is a function from $(\mathsf{X}, \mathscr{B})$ to $[0,1]$ such that $B \mapsto \Pi(x, B)$ is a probability measure on $(\mathsf{X}, \mathscr{B})$ for each $x \in \mathsf{X}$, and $x \mapsto \Pi(x, B)$ is $\mathscr{B}$-measurable for each $B \in \mathscr{B}$. The process $\{X_t\}$ satisfies $\mathbb{P}\{X_{t+1} \in B \mid X_t = x\} = \Pi(x, B)$ for all x in X and $B \in \mathscr{B}$.

conditioning on the event $X_0 = x$, so that, for any $h \in L_1(\pi)$ and any $n \in \mathbb{N}$, we have

$$\mathbb{E}_x h(X_n) = \int h(x)\Pi^n(x, \mathrm{d}y). \tag{37}$$

Also, for convenience, we set

$$\hat{g}(x) := (Vg)(x) = \int\int \phi(x, y, \eta) g(x, y, \eta) \nu(\mathrm{d}\eta)\Pi(x, \mathrm{d}y).$$

Lemma C.2. *For any $h \in \mathcal{H}_p$ and all $x \in \mathsf{X}$ we have*

$$V^n h(x) = \mathbb{E}_x \prod_{i=1}^{n} \Phi_i \, h(X_n). \tag{38}$$

Proof. Equation (38) holds when $n = 1$ because

$$Vh(x) = \int\int \phi(x, y, \eta)\nu(\mathrm{d}\eta)h(y)\Pi(x, \mathrm{d}y) = \mathbb{E}_x \, \Phi_1 h(X_1).$$

Now suppose (38) holds at arbitrary $n \in \mathbb{N}$. We claim it also holds at $n+1$. Indeed,

$$V^{n+1}h(x) = \mathbb{E}_x \, \Phi_1 V^n h(X_1) = \mathbb{E}_x \, \Phi_1 \, \mathbb{E}_{X_1} \prod_{i=2}^{n+1} \Phi_i \, h(X_{n+1}) = \mathbb{E}_x \, \mathbb{E}_{X_1} \prod_{i=1}^{n+1} \Phi_i \, h(X_{n+1}).$$

An application of the law of iterated expectations completes the proof. □

Lemma C.3. *For each $h \in \mathcal{H}_p$, $x \in \mathsf{X}$ and $n \in \mathbb{N}$ we have*

$$V^n h(x) = 0 \implies \int h(y)\Pi^n(x, \mathrm{d}y) = 0.$$

Proof. Fix $h \in \mathcal{H}_p$, $x \in \mathsf{X}$ and $n \in \mathbb{N}$ with $V^n h(x) = 0$. It follows from Lemma C.2 that $\mathbb{E}_x \prod_{i=1}^n \Phi_i \, h(X_n) = 0$, which in turn implies that $\prod_{i=1}^n \Phi_i \, h(X_n) = 0$ holds $\mathbb{P}_x$-a.s. But then, by the positivity in Assumption 2.1, $h(X_n) = 0$ holds $\mathbb{P}_x$-a.s. Hence $\mathbb{E}_x h(X_n) = 0$. By (37), this is equivalent to $\int h(y)\Pi^n(x, \mathrm{d}y) = 0$. □

Lemma C.4. *If $h \in \mathcal{H}_p$ with $h \gg 0$, then $V^n h \gg 0$ for all $n \in \mathbb{N}$.*

Proof. It suffices to show this is true when $n = 1$, after which we can iterate. To this end, fix $h \in \mathcal{H}_p$ with $h > 0$ on $B \in \mathscr{B}$ with $\pi(B) = 1$. Suppose that

$$Vh(x) = \int h(y) \left[\int \phi(x, y, \eta)\nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}y) = 0.$$

Since ϕ is positive, we must then have $\Pi(x, B) = 0$. But π is invariant, so $\pi(B) = \int \Pi(x, B)\pi(\mathrm{d}x) = 0$. Contradiction. □

Lemma C.5. *The valuation operator V is irreducible on $L_p(\pi)$.*

Proof. Suppose to the contrary that there exists a closed ideal J in $L_p(\pi)$ such that V is invariant on J and J is neither $\emptyset$ nor $L_p(\pi)$ itself. Since J is a closed ideal in $L_p(\pi)$, there exists a set $B \in \mathscr{B}$ such that $J = \{f \in L_p(\pi) : f = 0 \; \pi\text{-a.e. on } B\}$.[22] Moreover, since J is neither empty nor the whole space, it must be that, for this set B that defines J, we have $0 < \pi(B) < 1$.

Because V is invariant on J, we have $V^n h \in J$ for all $h \in J$ and $n \in \mathbb{N}$. In particular, $V^n \mathbb{1}_{B^c}$ is in J for all $n \in \mathbb{N}$. This means that $V^n \mathbb{1}_{B^c}(x) = 0$ for π-almost all $x \in B$ and all n in $\mathbb{N}$. Fixing an $x \in B$ and applying Lemma C.3, we then have $\Pi^n(x, B^c) = 0$ for all $n \in \mathbb{N}$. But $\pi(B) < 1$, so $\pi(B^c) > 0$. This contradicts irreducibility of the stochastic kernel Π, which in turn violates Assumption 2.2. $\square$

The following is a local spectral radius result suitable for $L_p(\pi)$ that draws on Zabreiko et al. (1967) and Krasnosel'skii et al. (2012).[23] The proof provided here is due to Mirosława Zima (private communication). In the statement of the theorem, a quasi-interior element of the positive cone of a Banach lattice $\mathcal{E}$ is a nonnegative element h satisfying $\langle h, g \rangle > 0$ for any nonzero element of the positive cone of the dual space $\mathcal{E}^*$. (See Krasnosel'skii et al. (2012) for more details.)

Theorem C.6. *Let h be an element of a Banach lattice $\mathcal{E}$ and let M be a positive and compact linear operator. If h is quasi-interior, then $\|M^n h\|^{1/n} \to r(M)$ as $n \to \infty$.*

Proof. Let h and M be as in the statement of the theorem and let $\mathcal{E}_+$ be the positive cone of $\mathcal{E}$. Let $r(h, M) := \limsup_{n \to \infty} \|M^n h\|^{1/n}$. From the definition of $r(M)$ it is clear that $r(h, M) \leqslant r(M)$. To see that the reverse inequality holds, let λ be a constant satisfying $\lambda > r(h, M)$ and let

$$h_\lambda := \sum_{n=0}^{\infty} \frac{M^n h}{\lambda^{n+1}}. \tag{39}$$

The point h_λ is a well-defined element of $\mathcal{E}_+$ by $\limsup_{n \to \infty} \|M^n h\|^{1/n} < \lambda$ and Cauchy's root test. It is also quasi-interior, since the sum in (39) includes the quasi-interior element h, and since M maps $\mathcal{E}_+$ into itself. Moreover, by standard Neumann series theory (e.g., Krasnosel'skii et al. (2012), theorem 5.1), the point h_λ also

[22] See, for example, Gerlach and Nittka (2012), p. 765.

[23] The result suits $L_p(\pi)$ because it allows the interior of the positive cone to be empty.

has the representation $h_\lambda = (\lambda I - M)^{-1}h$, from which we obtain $\lambda h_\lambda - Mh_\lambda = h$. Because $h \in \mathcal{E}_+$, this implies that $Mh_\lambda \leqslant \lambda h_\lambda$. Applying this last inequality, compactness of M, quasi-interiority of h_λ and theorem 5.5 (a) of Krasnosel'skii et al. (2012), we must have $r(M) \leqslant \lambda$. Since this inequality was established for an arbitrary λ satisfying $\lambda > r(h, M)$, we conclude that $r(h, M) \geqslant r(M)$.

We have shown that $\limsup_{n\to\infty} \|M^n h\|^{1/n} = r(M)$. Since M is compact, Corollary 1 of Daneš (1987) gives $\limsup_{n\to\infty} \|M^n h\|^{1/n} = \lim_{n\to\infty} \|M^n h\|^{1/n}$. □

Theorem C.7. *The growth exponent $\mathcal{L}^p_\Phi$ satisfies $\exp(\mathcal{L}^p_\Phi) = r(V)$, where $r(V)$ is the spectral radius of V in $L_p(\pi)$.*

Proof. Let $\mathbb{1} = \mathbb{1}_\mathsf{X} \equiv 1$ and let $\|\cdot\|$ be the norm in $L_p(\pi)$. By Lemma C.2, we have $V^n\mathbb{1}(x) = \mathbb{E}_x \prod_{i=1}^n \Phi_i$. Using this and the definition of $\mathcal{L}^p_\Phi$ in (9), we have

$$\exp(\mathcal{L}^p_\Phi) = \lim_{n\to\infty} \left\{ \mathbb{E}\left[\mathbb{E}_x \prod_{t=1}^n \Phi_t\right]^p \right\}^{1/(np)} = \lim_{n\to\infty} \frac{1}{n} \ln \|V^n \mathbb{1}\|.$$

As a consequence, it suffices to show that

$$\lim_{n\to\infty} \|V^n\mathbb{1}\|^{1/n} = r(V). \tag{40}$$

In doing so, we aim to apply Theorem C.6. We cannot do so directly because V is not compact. However, by Assumption 2.3 we can choose an $i \in \mathbb{N}$ such that V^i is a compact linear operator on $L_p(\pi)$. Fix $j \in \mathbb{N}$ with $0 \leqslant j \leqslant i-1$. By Lemma C.4 we know that $V^j\mathbb{1}$ is positive π-almost everywhere on X, and is therefore quasi-interior.[24] As a result, Theorem C.6 applied to V^i with initial condition $h := V^j\mathbb{1}$ yields

$$\|V^{in}V^j\mathbb{1}\|^{1/n} = \|V^{in+j}\mathbb{1}\|^{1/n} \to r(V^i) \qquad (n\to\infty).$$

But $r(V^i) = r(V)^i$, so $\|V^{in+j}\mathbb{1}\|^{1/(in)} \to r(V)$ as $n \to \infty$. It follows that

$$\|V^{in+j}\mathbb{1}\|^{1/(in+j)} \to r(V).$$

As this is shown to be true for any integer j satisfying $0 \leqslant j \leqslant i-1$, we can conclude that (40) is valid. □

[24] By the Riesz Representation Theorem, the dual space of $L_p(\pi)$ is isometrically isomorphic to $L_q(\pi)$ where $1/p + 1/q = 1$. If g is a nonnegative and nonzero element of $L_q(\pi)$ then it is positive on a set of positive π measure. Since $f \gg 0$ on X, the produce fg must be positive on a set of positive π measure. Hence $\int fg \,\mathrm{d}\pi > 0$, so f is quasi-interior.

To prove Theorem 2.1, we will also need the following two lemmas:

Lemma C.8. *The equilibrium price operator T is a self-map on $\mathcal{H}_p$. It has a fixed point in $\mathcal{H}_p$ if and only if there exist elements h_0, h in $\mathcal{H}_p$ such that $T^n h_0 \to h$ as $n \to \infty$.*

Proof. To see that T is a self-map on $\mathcal{H}_p$, fix $h \in \mathcal{H}_p$ and recall from (7) that $Th(x) = Vh(x) + \mathbb{E}_x \, \Phi_{t+1} G_{t+1}$. The fact that V maps $\mathcal{H}_p$ to itself, which is implied by Assumption 2.3, combined with Minkowski's inequality, means we need only prove that the function $m(x) := \mathbb{E}_x \, \Phi_{t+1} G_{t+1}$ is in $\mathcal{H}_p$. This will be true if $m(X_t)$ has finite p-th moment under $\mathbb{E}$. By Jensen's inequality and the law of iterated expectations, it suffices to show that $\mathbb{E} \, (\Phi_{t+1} G_{t+1})^p < \infty$, which is true by the moment condition in Assumption 2.3.

To prove the second claim in Lemma C.8, we suppose first that there exist h_0, h in $\mathcal{H}_p$ such that $T^n h_0 \to h$ as $n \to \infty$. Since T maps f into $Vf + \hat{g}$ and V is a bounded linear operator on $L_p(\pi)$, we know that T is continuous as a self-map on $L_p(\pi)$. Letting $h_n = T^n h_0$, we have $h_n \to h$ and hence, by continuity, $Th_n \to Th$. But, by the definition of the sequence $\{h_n\}$, we must also have $Th_n \to h$. Hence $Th = h$.

Conversely, if T has a fixed point $f \in \mathcal{H}_p$, then the condition in the statement of Lemma C.8 is satisfied with $h_0 = h = f$. □

Proposition C.9. *If T has a fixed point in $\mathcal{H}_p$, then $\mathcal{L}_\Phi^p < 0$.*

Proof. Let V^* be the adjoint operator associated with V. Since V is irreducible (see Lemma C.5) and V^i is compact for some i, the version of the Krein–Rutman theorem presented in lemma 4.2.11 of Meyer-Nieberg (2012) together with the Riesz Representation Theorem imply existence of an e^* in the dual space $L_q(\pi)$ such that

$$e^* \gg 0 \quad \text{and} \quad V^* e^* = r(V) e^*. \tag{41}$$

Let h be a fixed point of T in $\mathcal{H}_p$. Clearly h is nonzero, since $T0 = V0 + \hat{g} = \hat{g}$ and $\hat{g}$ is not the zero function (see Assumption 2.1). Moreover, since h is a fixed point, we have $h = Vh + \hat{g}$ and hence, with the inner production notation $\langle \phi, f \rangle := \int \phi f \, \mathrm{d}\pi$,

$$\langle e^*, h \rangle = \langle e^*, Vh \rangle + \langle e^*, \hat{g} \rangle = \langle V^* e^*, h \rangle + \langle e^*, \hat{g} \rangle = r(V) \langle e^*, h \rangle + \langle e^*, \hat{g} \rangle.$$

In other words,

$$(1 - r(V)) \langle e^*, h \rangle = \langle e^*, \hat{g} \rangle.$$

Both h and $\hat{g}$ are nonzero in $L_p(\pi)$ and e^* is positive π-a.e., so $\langle e^*, h\rangle > 0$ and $\langle e^*, \hat{g}\rangle > 0$. It follows that $r(V) < 1$. By Theorem C.7, we have $\mathcal{L}^p_\Phi = \ln r(V)$, which proves the claim in the lemma. □

Proof of Theorem 2.1. By Lemma C.8, (b) and (c) of Theorem 2.1 are equivalent, so it suffices to show that (d) $\implies$ (c) $\implies$ (a) $\implies$ (d). Of these, the implications (d) $\implies$ (c) is trivial, and (c) $\implies$ (a) was established in Proposition C.9. Hence we need only show that (a) $\implies$ (d).

To see that (a) implies (d), suppose that $\mathcal{L}^p_\Phi < 0$. Then, by Theorem C.7, we have $r(V) < 1$. Using Gelfand's formula for the spectral radius, which states that $r(V) = \lim_{n\to\infty} \|V^n\|^{1/n}$ with $\|\cdot\|$ as the operator norm, we can choose $n \in \mathbb{N}$ such that $\|V^n\| < 1$. Then, for any $h, h' \in \mathcal{H}_p$ we have

$$\|T^n h - T^n h'\| = \|V^n h - V^n h'\| = \|V^n(h - h')\| \leqslant \|V^n\| \cdot \|h - h'\|.$$

Observe that $\mathcal{H}_p$ is closed in $L_p(\pi)$, since $L_p(\pi)$ is a Banach lattice. Hence $\mathcal{H}_p$ is complete in the norm topology. Existence, uniqueness and global stability now follow from a well-known extension to the Banach contraction mapping theorem (see, e.g., p. 272 of Wagner (1982)).

Lastly, to see that (11) holds, suppose that (a)–(d) are true. Then $r(V) < 1$, which implies that $(I - V)^{-1}$ is well-defined on $\mathcal{H}_p$ and equals $\sum_{i=0}^\infty V^i$ (see, e.g., theorem 2.3.1 and corollary 2.3.3 of Atkinson and Han (2009)). In particular, the fixed point of T is given by $h^* = \sum_{n=0}^\infty V^n \hat{g}$. Applying (38) to this sum verifies the claim in (11). □

Proof of Proposition 3.1. Fix $p \geqslant 1$. If Assumptions 2.1–2.2 hold and X is a finite set endowed with the discrete topology, then all functions from X to $\mathbb{R}$ are measurable and have finite p-th moment, so $L_p(\mathsf{X}, \mathbb{R}, \pi) = \mathbb{R}^{\mathsf{X}}$ and $\mathcal{H}_p = \mathbb{R}^{\mathsf{X}}_+$. It follows that $\hat{g} \in \mathcal{H}_p$ and V is a bounded linear operator from $L_p(\mathsf{X}, \mathbb{R}, \pi)$ to itself (since every linear operator mapping a finite dimensional normed vector space to itself is bounded). By the Heine–Borel theorem, bounded subsets in finite dimensional space have compact closure, so V is also (eventually) compact. Thus, Assumption 2.3 holds. Finally, $\mathcal{L}^p_\Phi = \mathcal{L}^1_\Phi$ by the identity in (12), since, in a finite dimension normed linear space, the spectral radius is independent of the choice of norm (due to equivalence of norms combined with Gelfand's formula for the spectral radius). □

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