Stability of Equilibrium Asset Pricing Models: 
A Necessary and Sufficient Condition

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\textbf{ABSTRACT.} We obtain an exact necessary and sufficient condition for the existence and uniqueness of equilibrium asset prices in infinite horizon, discrete-time, arbitrage free environments. Through several applications we show how the condition sharpens and improves on previous results. We connect the condition, and hence the problem of existence and uniqueness of asset prices, with the recent literature on stochastic discount factor decompositions. Finally, we discuss computation of the test value associated with our condition, providing a Monte Carlo method that is naturally parallelizable.

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\section{1. INTRODUCTION}

One fundamental problem in economics is the pricing of an asset paying a stochastic cash flow with no natural termination point, such as a sequence of dividends.

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In discrete-time no-arbitrage environments, the equilibrium price process \( \{P_t\}_{t \geq 0} \) associated with a dividend process \( \{D_t\}_{t \geq 1} \) obeys

\[
P_t = \mathbb{E}_t M_{t+1}(P_{t+1} + D_{t+1}) \quad \text{for all } t \geq 0, \tag{1}
\]

where \( \{M_t\}_{t \geq 1} \) is the pricing kernel or one-period stochastic discount factor process of a representative investor.\(^1\) Two questions immediately arise in connection with these dynamics:

1. Given \( \{D_t, M_t\}_{t \geq 1} \), does there exist a unique equilibrium price process?
2. How can we characterize and evaluate such prices whenever they exist?

Although these questions have been the subject of intensive analysis in the past, the number of settings where we lack a clear picture is rising rather than falling. The main reason is that models of dividend processes and state price deflators have become more sophisticated in recent years, in an ongoing effort to better match financial data and resolve outstanding puzzles in the literature (see, e.g., Campbell and Cochrane (1999), Barro (2006), or Bansal and Yaron (2004) and subsequent iterations of these models). This complexity makes questions 1–2 increasingly challenging to address, especially in quantitative applications with discount rates that are close to the growth rates of underlying cash flows.\(^2\) In general there have been few sufficient conditions proposed that (a) imply existence and uniqueness of equilibrium prices, (b) are weak enough to be useful in modern quantitative analysis, and (c) are practical to implement in applied settings.

To address this absence, we introduce a condition for existence and uniqueness of equilibria that is both weak enough to hold in realistic applications—in fact necessary as well as sufficient, and hence as weak as possible—and practical in the sense that testing the condition focuses on a single value. The value in question is the stability exponent

\[
\mathcal{L}_M := \lim_{n \to \infty} \frac{\ln \psi_n}{n}, \tag{2}
\]

\(^1\)See, for example, Rubinstein (1976), Ross (1978), Kreps (1981), Hansen and Richard (1987) or Duffie (2001). Here and below, prices are on ex-dividend contracts. Cum-dividend contracts are a simple extension to what follows.

\(^2\)Throughout this paper, we consider only fundamental solutions to the asset pricing problem (1), setting aside rational bubbles (see, e.g., Santos and Woodford (1997)).
where $\psi_n := \mathbb{E} \prod_{t=1}^{n} M_t$ is the price of a zero-coupon default-free bond with maturity $t$, and the expectation averages over possible draws of the time zero state. The value $L_M$ corresponds to the asymptotic growth rate of the average price process $\{\psi_t\}$, since existence of the limit in (2) implies that

$$\ln \frac{\psi_{t+1}}{\psi_t} \approx L_M \quad \text{for large } t.$$  \hfill (3)

In a standard setting with uncertainty driven by an exogenous and time homogeneous first order Markov state process, we show that, for the case where dividends are stationary and have finite first moment, existence and uniqueness of an equilibrium price process $\{P_t\}$ satisfying (1) is exactly equivalent to the statement $L_M < 0$. When dividends are nonstationary we replace $L_M$ with an analogous quantity for a growth adjusted SDF process and obtain a parallel existence and uniqueness result for the price-dividend ratio.

In addition to these existence and uniqueness results, we also study a method for computing equilibrium prices (or price-dividend ratios) for the dividend process $\{D_t\}$ using successive approximations via an equilibrium price operator. We show that this algorithm is globally convergent if and only if $L_M < 0$. In other words, the negative growth condition necessary and sufficient for existence and uniqueness is also necessary and sufficient for global convergence of successive approximations. One interesting implication is that convergence of the algorithm itself implies existence and uniqueness of equilibrium asset prices.

Regarding the intuition behind our result, recall that $L_M$ is the asymptotic growth rate for the average price $\psi_t$. If we consider the graph of $t \mapsto -\psi_t$ as the average default-free yield curve, then the condition $L_M < 0$ means that yields are asymptotically positive, as are forward rates in (3). This indicates a fundamental preference for current payoffs over future payoffs, on average, over the long run, which generates finite, well defined prices for infinite horizon cash flows. This is the sufficiency component of our existence result. It confirms the standard intuition and its value is largely technical, in the sense that the set of models it covers is particularly broad.

More striking and remarkable is the fact that this yield curve condition is necessary as well, and hence exactly characterizes the set of models with well defined equilibrium prices. This result rests on irreducibility of the underlying state process that supplies persistent stochastic components to dividends and the stochastic
discount factor—a condition that holds in all applications we consider. The necessity argument is built on a “local spectral radius” result for positive operators in Banach lattices due to Zabreiko et al. (1967) and Forster and Nagy (1991). Using this result, we show that, for any positive cash flow with finite first moment, the asymptotic mean growth rate of its discounted payoff stream is equal to the principal eigenvalue of an associated valuation operator. When a regularity condition on this operator is satisfied, this in turn is equal to the exponential of $L_M$. If the principal eigenvalue equals or exceeds unity and the state process is irreducible, then the sum of expected discounted payoffs grows without bound.

As stated above, in the environment we consider, the condition $L_M < 0$ also gives uniqueness of equilibrium prices whenever they exist. One way to understand this is to view $M_{t+1}$ in (1) as a random “contraction factor” around which a contraction mapping argument can be built, looking forwards in time. The operator in this argument has, as its fixed point, an equilibrium price function, which maps states into equilibrium prices. If there exists a constant $\theta$ with $M_t \leq \theta < 1$ with probability one, then (1) implies this operator will be a contraction of modulus $\theta$, yielding existence of a unique equilibrium. However, in most applications, $M_{t+1} > 1$ holds on a set of positive probability, due to the fact that payoffs in bad states have high value. Thus, a direct one step contraction argument is problematic. We replace this with the weaker condition $L_M < 0$, which requires instead that $M_t < 1$ holds on average over the long run, and exploit some algebraic structure of the asset pricing model to show that such a property is in fact sufficient.

As an additional contribution to this line of analysis, we discuss methods for calculating the growth rate $L_M$ in those cases where no analytical solution exists. First, we show that, when the state space is finite, $L_M$ can be calculated using a familiar method for computing spectral radii via numerical linear algebra. Second, we propose a Monte Carlo method that involves simulating independent paths for the discount factor process and averaging to produce the expectation in (1). This method is inherently parallelizable and suited to settings where the state space is large.

As one illustration of the method, we consider a model of asset prices with Epstein–Zin recursive utility, multivariate cash flows and time varying volatility studied in Schorfheide et al. (2018), which in turn builds on the long run risk framework developed by Bansal and Yaron (2004). Hitherto no results have been available on
existence and uniqueness of equilibria in the underlying theoretical model, partly
because Schorfheide et al. (2018) and other related studies have focused their attention on approximations generated using perturbation methods. While these studies have been insightful, Pohl et al. (2018) recently demonstrated the importance of nonlinearities embedded in the original model for determining asset prices. We focus on the original model and show that $L_M < 0$ holds for at the benchmark parameterization. This indicates existence of a unique set of equilibrium prices, along with a globally convergent method of computing them. The fact that our conditions are necessary as well as sufficient allows us to examine how far this positive result can be pushed as we shift parameters relative to the benchmark.

While the above result is new, our main theorem also generalizes existing results on existence and uniqueness of asset prices obeying the dynamics in (1) over an infinite horizon. For example, when consumption and dividends are driven by a finite state Markov process, it is well known that existence and uniqueness of equilibrium prices holds whenever the spectral radius of a certain valuation matrix is less than unity (see, e.g., Mehra and Prescott (1985)). We show that, for such problems, the log of the spectral radius in question is equal to $L_M$, so this standard result is a special case of our main theorem. Since our results are necessary as well as sufficient under irreducibility of the state process, they further extend our understanding of the finite state case.

We also encompass and extend the existence and uniqueness of Lucas (1978), who studied a model with infinite state space and SDF of the form

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}. \quad (4)$$

Here $\{C_t\}$ is a consumption process, $\beta$ is a state independent discount component and $u$ is a period utility function. Using a change of variable, Lucas (1978) obtains a modified pricing operator with contraction modulus equal to $\beta$, and hence, by Banach’s contraction mapping principle, a unique equilibrium price process. We show below that his theorem is a special case of our main theorem result.

While Lucas (1978) frames his contraction based results in a space of bounded functions, our analysis admits unbounded solutions. This is achieved by embedding the equilibrium problem in a space of candidate solutions with finite first
moments. Such a setting is arguably more natural for the study of forward looking stochastic sequences, since the forward looking restriction is itself stated in terms of expectations. Adopting this setting allows us to generalize the existence and uniqueness results for equilibrium prices obtained in Calin et al. (2005) and Brogueira and Schütze (2017), which extend Lucas (1978) by allowing for habit formation and unbounded utility.\textsuperscript{3}

Some of the preceding results are analytical, based on exact expressions for the exponent $L_M$, while others, such as the treatment of the asset pricing problem in Schorfheide et al. (2018), rely on numerical evaluation of $L_M$ due to complexity of the dividend and SDF processes. Of course one might object to a numerical test for existence and uniqueness of equilibria, since such tests introduce rounding or discretization errors, which, in the worst case, can qualitatively affect results. However, we find that some modern quantitative asset pricing studies are too complex—and too close to the boundary between stability and instability—to allow for successful use of analytically tractable sufficient conditions. We show that such restrictions are typically violated in practice, even at parameterizations where the asset pricing models in question do have unique and well defined equilibria.

Our work is also connected to the literature on stochastic discount factor decompositions found in Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Hansen (2012); Borovička et al. (2016); Christensen (2017); Qin and Linetsky (2017) and other recent studies. These decompositions are used to extract a permanent growth component and a martingale component from the stochastic discount process, with the rate in the permanent growth component being driven by the principal eigenvalue of the valuation operator associated with stochastic discount factor.

We show that the log of this principal eigenvalue is equal to $L_M$ in our setting, using the local spectral radius result discussed above. Unlike to the literature on

\textsuperscript{3}Not surprisingly, our results also generalize the simple risk neutral case $M_t \equiv \beta$, which is linear and hence easily treated by standard methods (see, e.g., Blanchard and Kahn (1980)). The existence of a unique equilibrium when $\beta \in (0, 1)$ is a special case of our results because the $n$ period state price deflator is just $\beta^n$, so, by the definition of the exponent $L_M$ in (2), we have $L_M = \ln \beta$. The condition $\beta \in (0, 1)$ therefore implies $L_M < 0$ and hence existence of a unique solution.
stochastic discount factor decompositions, which uses the permanent growth component to shed light on the structure of valuation for payoffs at alternative horizons, our concern is with existence and uniqueness of equilibria over infinite horizons. Thus, we extend the reach of the existing theory by mapping the permanent growth component to an exact necessary and sufficient condition for these properties. Through this process we also offer new ways to compute the permanent growth component via its connection to the exponent $L_M$.

On a technical level, there is some overlap between this paper and the analysis of existence of Epstein–Zin utilities contained in Borovička and Stachurski (2017). We use some common results local spectral radius methods and Monte Carlo methods. However, the topic is different and so are the essential functional equations.

Regarding the mathematical literature, the asymptotic growth rate $L_M$ of the zero coupon bond price with maturity $t$ can alternatively understood as a “forward Lyapunov exponent,” due to its connection with the integrated Lyapunov exponent (see, e.g., Furstenberg and Kesten (1960) or Knill (1992)), which, for the pricing kernel process $\{M_t\}$, takes the form

$$I_M := \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \ln M_t. \quad (5)$$

When $\{M_t\}$ is stationary, this reduces to $\mathbb{E} \ln M_t$, which is considerably simpler than the forward Lyapunov exponent $L_M$. However, it is immediate from Jensen’s inequality that $I_M \leq L_M$, and, as $L_M < 0$ is necessary and sufficient for existence and uniqueness, the inequality $I_M < 0$ is necessary but not sufficient.

The failure of the condition $I_M < 0$ to generate existence is due to the fact that it takes into account only the marginal distribution of $M_t$. This is not enough because stability requires that we rule out long epochs during which the SDF exceeds unity. In other words, we must control persistence in the SDF process, which requires restrictions on the full joint distribution. More generally, the reason that the integrated Lyapunov exponent is not suitable for studying the asset pricing equation (1) is that traditional Lyapunov theory was developed for backward looking equations, whereas (1) is forward looking.

The rest of the paper is structured as follows: Our main theoretical result is presented in section 2. Section 3 treats applications and section 4 extends the theory. Section 5 concludes. A discussion of numerical methods for implementing
our test can be found in appendix A. All proofs are deferred until appendix B. Computer code that replicates our numerical results and figures can be found at https://github.com/jstac/asset_pricing_code.

2. A Necessary and Sufficient Condition

In this section we set up our framework in detail and state our main results.

2.1. Environment. We will work with the generic forward looking model

\[ Y_t = \mathbb{E}_t [\Phi_{t+1} (Y_{t+1} + G_{t+1})] \quad \text{for all } t \geq 0. \]  

(6)

Here \( \{\Phi_t\} \) and \( \{G_t\} \) are given and \( \{Y_t\} \) is endogenous. The equilibrium pricing equation (1) is obviously one special case of (6). While \( \{\Phi_t\} \) and \( \{G_t\} \) will have different interpretations in other applications, it is convenient to refer to \( \{\Phi_t\} \) as the stochastic discount factor and \( \{G_t\} \) as the cash flow.

Example 2.1. Aside from the pricing equation (1), one common version of (6) arises when dividend growth is stationary, rather than dividends themselves. In this case we divide (1) by \( D_t \), which yields

\[ Q_t = \mathbb{E}_t \left[ M_{t+1} \frac{D_{t+1}}{D_t} (Q_{t+1} + 1) \right] \quad \text{when } Q_t := \frac{P_t}{D_t}. \]  

(7)

The price-dividend ratio \( Q_t \) is the endogenous process to be obtained. This maps to (6) when \( G_t \equiv 1 \) and \( \Phi_{t+1} = M_{t+1} D_{t+1} / D_t \).

We say that a stochastic process \( \{Y_t\} \) solves (6) if, with probability one, each \( Y_t \) is finite and (6) holds for all \( t \geq 0 \). To obtain a solution we require some auxiliary conditions on the state process, the cash flow and the stochastic discount process. The first of these is as follows:

Assumption 2.1. \( \Phi_t \) is a positive random variable and \( G_t \) is nonnegative and non-trivial in the sense that \( G_t > 0 \) on a set of positive probability.

Neither of these assumptions cost any generality. Positivity of \( \Phi_t \) is equivalent to assuming no arbitrage in a complete market setting.\(^4\) The assumption that \( G_t \) is nontrivial is also innocuous, since a trivial cash flow implies that \( Y_t = 0 \) for all \( t \).

\(^4\)See, for example, Hansen and Richard (1987), lemma 2.3.
To introduce the possibility of stationary Markov solutions, we assume that \( \{ \Phi_t \} \) and \( \{ G_t \} \) admit the representations

\[
\Phi_{t+1} = \phi(X_t, X_{t+1}, \eta_{t+1}) \quad \text{and} \quad G_{t+1} = g(X_t, X_{t+1}, \eta_{t+1})
\]

where \( \{ X_t \} \) is an underlying \( X \)-valued state process, \( \{ \eta_t \} \) is a \( W \)-valued innovation sequence and \( \phi \) and \( g \) are positive Borel measurable maps on \( X \times X \times W \). The sets \( X \) and \( W \) may be finite, measurable subsets of \( \mathbb{R}^n \), or infinite dimensional.\(^5\) The representations in (8) replicate the general multiplicative functional specifications considered in Hansen and Scheinkman (2009) and Hansen (2012) and are sufficient for all problems we consider.

The process \( \{ X_t \} \) is defined on some underlying probability space \((\Omega, \mathcal{F}, P)\), as is the innovation process \( \{ \eta_t \} \). The innovation process is assumed to be IID and independent of \( \{ X_t \} \). Each \( \eta_t \) has common distribution \( \nu \). The state process is assumed to be stationary and Markovian on \( X \). The common marginal distribution of each \( X_t \) is denoted by \( \pi \). The conditional distribution of \( X_{t+1} \) given \( X_t = x \) is denoted by \( \Pi(x, dy) \).

**Assumption 2.2.** The state process \( \{ X_t \} \) is irreducible.

This means that, regardless of the initial condition, subsets of the state space with positive probability under \( \pi \) are visited eventually with positive probability.\(^6\) The assumption is satisfied in all applications we consider. Settings where assumption 2.2 fails can usually be rectified by appropriately minimal choice of the state space.

A measurable function \( h \) from \( X \) to \( \mathbb{R} \) is called a *Markov solution* to the forward looking equation (6) if the process \( \{ Y_t \} \) defined by \( Y_t = h(X_t) \) for all \( t \) is a solution to (6). Inserting \( Y_i = h(X_i) \) for all \( i \) into \( Y_t = \mathbb{E}_t \Phi_{t+1}(Y_{t+1} + G_{t+1}) \) and conditioning on \( X_t = x \), we see that \( h \) will be a Markov solution if

\[
h = Vh + \hat{g} \quad \text{where} \quad \hat{g}(x) := \int \int \phi(x, y, \eta)g(x, y, \eta)\nu(d\eta)\Pi(x, dy)
\]

\(^5\)We assume only that \( X \) and \( W \) are separable and completely metrizable topological spaces. See section B for details.

\(^6\) More formally, the definition is that, for each Borel set \( B \subset X \) with \( \pi(B) > 0 \) and each \( x \in X \), there exists an \( n \in \mathbb{N} \) such that \( \Pi^n(x, B) > 0 \). Here \( \Pi^n \) represents \( n \) step transition probabilities and is defined recursively by \( \Pi^1 = \Pi \) and \( \Pi^n(x, B) = \int \Pi^{n-1}(x, dz)\Pi(z, B) \). See, for example, Meyn and Tweedie (2009), ch. 4.
and $V$ is the *valuation operator* defined by

$$Vh(x) := \int h(y) \left[ \int \phi(x, y, \eta) v(d\eta) \right] \Pi(x, dy).$$

(9)

The quantity $Vh(x)$ is interpreted as the present discounted value of payoff $h(X_{t+1})$ conditional on $X_t = x$. Letting $T$ represent the *equilibrium price operator* defined at Borel measurable function $h: X \to \mathbb{R}_+$ by

$$Th = Vh + \hat{g},$$

(10)

it is clear that $h$ is a Markov solution if and only if $h$ is a fixed point of $T$. We write $T^n$ for the $n$-th composition of $T$ with itself.

Before stating results, we need a candidate space for Markov solutions. To this end, for each $p \geq 0$, we let $L_p(X, \mathbb{R}, \pi)$ denote, as usual, the set of Borel measurable real-valued functions $h$ defined on the state space $X$ such that $\int |h(x)|^p \pi(dx)$ is finite. Let $\mathcal{H}_p$ be all nonnegative functions in $L_p(X, \mathbb{R}, \pi)$. In other words, $\mathcal{H}_p$ is the set of all nonnegative functions on $X$ such that $h(X_t)$ has finite $p$-th moment. This will be our candidate space, so if $p = 2$, say, then we seek solutions with finite second moment.\footnote{In what follows, all notions of convergence refer to standard norm convergence in $L_p$. As usual, functions equal $\pi$-almost everywhere are identified. Appendix B gives more details.}

Assumption 2.3. There exists a $p \geq 1$ such that $\hat{g} \in \mathcal{H}_p$ and $V$ is eventually compact as a linear operator from $L_p(X, \mathbb{R}, \pi)$ to itself.

The first part of assumption 2.3 is a finite moment restriction. The assumption is weakest when $p = 1$, and in fact this minimal restriction cannot be omitted, since the forward looking restriction is stated in terms of expectations. Such a restriction is not well defined without finiteness of first moments. On the other hand, we might wish to choose $p$ to be larger when possible, in order to impose more structure on our solution (e.g., finiteness of second moments is necessary for many asymptotic results related to estimation).

The “eventually compact” part of assumption 2.3 is a regularity condition, the details of which are given in appendix B. Analogous conditions can be found in the
literature on eigenfunction decompositions of valuation operators (see, e.g., assumption 2.1 in Christensen (2017)). In section 4 we show that the sufficiency component of theorem 2.1 below continues to be valid when this regularity condition is dropped.

2.2. **Existence and Uniqueness.** In addition, we introduce the $p$-th order stability exponent of the SDF process $\{\Phi_t\}$ as

$$L_{\Phi}^p := \lim_{n \to \infty} \frac{1}{np} \ln \mathbb{E}\left\{ \mathbb{E}_x \prod_{t=1}^n \Phi_t \right\}^p. \quad (11)$$

This is a generalization of the (first order) stability exponent $L_{\Phi}$ introduced in (12). The simplest cases arise when $p = 1$, since by the Law of Iterated Expectations, we have $\mathbb{E}\mathbb{E}_x = \mathbb{E}$ and hence

$$L_{\Phi}^1 = L_{\Phi} := \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}\left\{ \mathbb{E}_x \prod_{t=1}^n \Phi_t \right\}. \quad (12)$$

As discussed in (2)–(3), when $\{\Phi_t\}$ is the discount factor process, the exponent $L_{\Phi}^1$ can be interpreted as the growth rate in $t$ of the expected price of a zero-coupon bond with maturity $t$. We show below that, in many important applications, we can concentrate all our attention on $L_{\Phi}$, since $L_{\Phi}^p = L_{\Phi}$ for all $p$.

**Theorem 2.1.** If assumptions 2.1–2.3 hold, then the limit in (11) exists and all of the following statements are equivalent:

(a) $L_{\Phi}^p < 0$.
(b) A Markov solution $h^*$ exists in $\mathcal{H}_p$.
(c) A unique Markov solution $h^*$ exists in $\mathcal{H}_p$.
(d) There exists an $h$ in $\mathcal{H}_p$ such that $T^nh$ converges to some limit in $\mathcal{H}_p$ as $n \to \infty$.
(e) A unique Markov solution $h^*$ exists in $\mathcal{H}_p$ and $T^nh$ converges to $h^*$ for every $h \in \mathcal{H}_p$.

Moreover, if one and hence all of (a)–(e) are true, then the unique Markov solution $h^*$ satisfies

$$h^*(x) = \sum_{n=1}^{\infty} \mathbb{E}_x \prod_{i=1}^n \Phi_i G_n. \quad \text{for all } x \in X. \quad (13)$$
Parts (a)–(c) of theorem 2.1 tell us that $\mathcal{L}_\Phi^p < 0$ is both necessary and sufficient for a unique Markov solution with finite $p$-th moment to exist, and, moreover, that existence itself fails (rather than uniqueness) when $\mathcal{L}_\Phi^p \geq 0$. Condition (d) is valuable from an applied perspective because it shows that if iteration with $T$ does converge from some starting point, then the limit is necessarily a Markov solution, and also the only Markov solution in $\mathcal{H}_p$. Part (e) provides a globally convergent algorithm for computing the unique equilibrium whenever $\mathcal{L}_\Phi^p < 0$ holds. The representation in (13) states that, when the solution exists, it equals the infinite sum of discounted payoffs.

The successive approximations method for computing the solution offered in (e) of theorem 2.1 should be compared with other methods for solving asset pricing models, such as perturbations and projections. A comprehensive discussion of computational techniques for asset pricing models is given in Pohl et al. (2018), with the results generally favorable to projections. The successive approximations algorithm from theorem 2.1 can be thought of either as a robust alternative or complementary in the following sense: While projection methods are fast, they almost always require tuning, as output can be sensitive to both the choice of basis functions and the solver used for the associated nonlinear equations. In such cases, the globally convergent successive approximations method can be employed to first compute the solution. Projection methods can then be tuned until they reliably reproduce it.

2.3. Connection to Spectral Radius Arguments. In appendix B we show that, when assumptions 2.1–2.3 hold, the value $\mathcal{L}_\Phi^p$ is always finite and

$$\mathcal{L}_\Phi^p = \ln r(V)$$

(14)

where $r(V)$ is the spectral radius of the valuation operator when regarded as a linear self-map on $L_p(\mathcal{X}, \mathcal{R}, \pi)$. This result is central to our necessity result in particular and useful for computation. The spectral radius $r(V)$ also appears in the literature on stochastic discount factor decompositions discussed in the introduction, since it equals the log of the principal eigenvalue of the valuation operator $V$, which in turn determines the permanent growth component of the stochastic discount factor. Theorem 2.1 and (14) show that this permanent growth component exactly determines the boundary between existence and nonexistence of equilibria.
The ideas behind (14) can be understood by considering the following line of argument. Suppose, for simplicity, that the state space \( X \) is finite. Let \( \Pi(x, y) \) represent the probability of transitioning from \( x \) to \( y \) in one step. Note that, for all \( x \) in \( X \) and all \( n \) in \( \mathbb{N} \), we have

\[
V^n_{1}(x) = \mathbb{E}_x \prod_{t=1}^{n} \Phi_t, \tag{15}
\]

where \( V \) is the valuation matrix

\[
V(x, y) = \left[ \int \phi(x, y, \eta) \nu(d\eta) \right] \Pi(x, y) \tag{16}
\]

corresponding to the valuation operator defined in (9). In particular, \( V^n_{1}(x) \) is element \( x \) of the column vector \( V^n_{1} \), where \( V^n \) is the \( n \)-th power of \( V \). The expectation \( \mathbb{E}_x \) conditions on \( X_0 = x \). The identity in (15) can be confirmed by induction (consider, for example, the case \( n = 1 \)) and the intuition is straightforward: applying \( V \) to a payoff vector yields a present discounted value per unit of dividend. Thus, \( V^n_{1}(x) \) is the present discounted value of a zero coupon default-free bond that matures in \( n \) periods, contingent on current state \( n \). Since \( \Phi_t \) is the (growth adjusted) stochastic discount factor, \( \mathbb{E}_x \prod_{t=1}^{n} \Phi_t \) gives the same value.\(^8\)

By (15) and the law of iterated expectations, we have

\[
\| V^n_{1} \| := \sum_{x \in X} V^n_{1}(x) \pi(x) = \mathbb{E} \prod_{t=1}^{n} \Phi_t, \tag{17}
\]

where \( \| \cdot \| \) is the \( L_1 \) vector norm defined by \( \| h \| = \sum_{x \in X} |h(x)| \pi(x) \). Gelfand’s formula tells us that \( \| V^n \|^{1/n} \to r(V) \) as \( n \to \infty \) whenever \( \| \cdot \| \) is a matrix norm, and this result can be modified to show that \( \| V^n_{1} \|^{1/n} \to r(V) \) also holds.\(^9\)

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\(^8\)Lemma B.2 in the appendix provides a proof of (15) in a general (i.e., finite or infinite) state setting.

\(^9\)This last convergence claim is a so-called “local spectral radius” result. The local spectral radius argument yielding \( \| V^n_{1} \|^{1/n} \to r(V) \) in the general case uses a theorem for positive operators in Banach lattices due to Zabreiko et al. (1967) and Forster and Nagy (1991), which is notable for the fact that it can handle Banach lattices where the positive cone has empty interior. The fact that \( \lim \) can be used instead of \( \lim \sup \) in the definition of \( \mathcal{L}^p_\Phi \) is based on a theorem of Daneš (1987). Appendix B provides a detailed treatment. The technical results listed in this footnote are essential to our proofs.
the last result with (17) gives

\[ \mathcal{L}_\Phi = \lim_{n} \frac{1}{n} \ln \left\{ \mathbb{E} \prod_{t=1}^{n} \Phi_t \right\} = \lim_{n} \ln \left\{ \| V^n \|^{1/n} \right\} = \ln r(V), \]

which confirms the claim in (14).

2.4. The Finite State Case. The problem treated in this paper simplifies when the state space is finite, which is important partly because some theoretical specifications are finite and partly because numerical implementations inevitably reduce computations onto a finite set of floating point numbers. The following result is key:

**Proposition 2.2.** If assumptions 2.1–2.2 hold and, in addition, the state space \( X \) is finite, then assumption 2.3 also holds and \( \mathcal{L}_\Phi^p = \mathcal{L}_\Phi \) for all \( p \geq 1 \). In particular (b)–(e) of theorem 2.1 all hold at every \( p \geq 1 \) if and only if \( \mathcal{L}_\Phi < 0 \).

One implication is that, in the finite state case, one can always work with the simpler exponent \( \mathcal{L}_\Phi^1 = \mathcal{L}_\Phi \), which is easier to calculate than \( \mathcal{L}_\Phi^p \) at \( p > 1 \). The proof of proposition 2.2 rests on the fact that all norms are equivalent in finite dimensional normed linear space.

3. Applications

We now turn to applications of theorem 2.1, showing how its results can be applied by testing the condition \( \mathcal{L}_\Phi^p < 0 \) in a range of settings. We begin with relatively simple cases, which illustrate the methodology, and then continue to more sophisticated settings with time-varying risk and non-additive intertemporal preferences.

3.1. Constant Volatility and Relative Risk Aversion. We first treat a simple setting where the stability exponent can be computed analytically. This provides intuition and a benchmark for testing numerical calculations (see appendix A for the latter). As in Lucas (1978), we set the SDF to the standard time separable form

\[ M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}, \] (18)
where \( \beta \in (0, 1) \) is a state independent discount factor. Agents have CRRA utility
\[
u(c) = \frac{c^{1-\gamma}}{1-\gamma}\] where \( \gamma \geq 0 \) and \( \gamma \neq 1 \). (19)

Dividends and consumption growth obey the constant volatility specification from section I.A of Bansal and Yaron (2004), which is
\[
\ln \left( \frac{D_{t+1}}{D_t} \right) = \mu_d + \varphi X_t + \sigma_d \xi_{t+1} \\
\ln \left( \frac{C_{t+1}}{C_t} \right) = \mu_c + X_t + \sigma_c \epsilon_{t+1} \\
X_{t+1} = \rho X_t + \sigma \eta_{t+1} \]
(20a, 20b, 20c)
Here \(-1 < \rho < 1\) and \(\{(\xi_t, \epsilon_t, \eta_t)\}\) is IID and standard normal in \(\mathbb{R}^3\). We solve for the price dividend ratio using (7), which means, by the discussion following that equation, that \(G_t = 1\) and
\[
\Phi_{t+1} = M_{t+1} \frac{D_{t+1}}{D_t} = \beta \exp \{ (\mu_d + \varphi X_t + \sigma_d \xi_{t+1}) - \gamma(\mu_c + X_t + \sigma_c \epsilon_{t+1}) \}. \quad \text{(21)}
\]
It follows that
\[
\prod_{i=1}^n \Phi_i = \beta^n \exp \left\{ n(\mu_d - \gamma \mu_c) + (\varphi - \gamma) \sum_{i=1}^n X_i + \sigma_d \sum_{i=1}^n \xi_i - \gamma \sigma_c \sum_{i=1}^n \epsilon_i \right\}.
\]
Using (20c), we then have
\[
\left( \mathbb{E}_x \prod_{i=1}^n \Phi_i \right)^p = \beta^n \exp(p a_n x + p b_n), \quad \text{(22)}
\]
where \(a_n := (\varphi - \gamma) \rho (1 - \rho^n) / (1 - \rho)\) and
\[
b_n := n(\mu_d - \gamma \mu_c) + \frac{(\varphi - \gamma)^2 s_n^2 + n \sigma_d^2 + n(\gamma \sigma_c)^2}{2}.
\]
Here \(s_n^2\) is the variance of \(\sum_{i=1}^n X_i\). The next step in calculating \(\mathcal{L}_\Phi^p\) is to take the unconditional expectation of (22), which amounts to integrating with respect to the stationary distribution \(\pi = N(0, \sigma^2 / (1 - \rho^2))\). This yields
\[
\mathbb{E} \left( \mathbb{E}_x \prod_{i=1}^n \Phi_i \right)^p = \beta^n \exp \left( \frac{(pa_n \sigma)^2}{2(1 - \rho^2)} + pb_n \right),
\]
and hence
\[
\mathcal{L}_\Phi = \lim_{n \to \infty} \left\{ \ln \beta + \frac{p}{n} \frac{(a_n \sigma)^2}{2(1 - \rho^2)} + \frac{b_n}{n} \right\} = \ln \beta + \lim_{n \to \infty} \frac{b_n}{n}, \quad \text{(23)}
\]
where the second equality uses the fact that \( a_n \) converges to a finite constant. Some algebra yields

\[
\frac{s_n^2}{n} = \frac{\sigma^2}{1 - \rho^2} \left\{ 1 + \frac{2(n - 1)}{n} \frac{\rho}{1 - \rho} - \frac{2\rho^2}{n} \cdot \frac{1 - \rho^{n-1}}{(1 - \rho)^2} \right\}. \tag{24}
\]

Combining this with (23), we find that

\[
\mathcal{L}_\Phi^p = \ln \beta + \mu_d - \gamma \mu_c + \frac{\sigma^2 (\phi - \gamma)^2}{2 (1 - \rho)^2} + \frac{\sigma_d^2 + (\gamma \sigma_c)^2}{2}. \tag{25}
\]

The value of \( \mathcal{L}_\Phi^p \) represents the long-run growth rate of the discounted dividend \( \Phi_t \). In expression (25), the term \( \mu_d + \sigma_d^2/2 + [\varphi \sigma/(1 - \rho)]^2/2 \) corresponds to the long-run dividend growth rate, \( \ln \beta - \gamma \mu_c + (\gamma \sigma_c)^2/2 + [\gamma \sigma/(1 - \rho)]^2/2 \) to the (negative of) the long-run discount rate, and \( \varphi \gamma \sigma^2/(1 - \rho)^2 \) is the long-run covariance between the two.\(^{10}\)

When do the conditions of theorem 2.1 hold? Since \( G_t = 1 \) and \( \Phi_t \) is given by (21), assumption 2.1 is clearly valid. The state process (20c) is certainly irreducible, so assumption 2.2 holds. Moreover, in view of (27), the valuation operator \( V \) has the form \( Vh(x) = \beta \exp \{ a x + b \} \int h(y) q(x, y) \, dy \) for suitably chosen constants \( a \) and \( b \), where \( q \) is the Gaussian transition density associated with (20c). From this expression it can be verified that assumption 2.3 holds at \( p = 2 \) via proposition B.1 in the appendix. Hence theorem 2.1 implies that a unique equilibrium price dividend ratio exists whenever parameters are such that the right hand side of (25) is negative.

3.2. A Finite State Application. In many asset pricing problems, the exogenous state evolves as a finite Markov chain.\(^{11}\) For concreteness, we maintain the setting of section 3.1, apart from switching the state process \( \{ X_t \} \) from the AR(1) dynamics in (20c) to a Markov chain taking values in finite set \( X \) and obeying stochastic transition matrix \( \Pi \). The pricing problem in (7) then reduces to solving for a \( q \)

\(^{10}\) Notice that \( \mathcal{L}_\Phi^p \) in (25) does not depend on \( p \). In particular, we have \( \mathcal{L}_\Phi^p = \mathcal{L}_\Phi := \mathcal{L}_\Phi \) for all \( p \). This matches the finding that \( \mathcal{L}_\Phi^p = \mathcal{L}_\Phi \) for all \( p \) in the finite dimensional case, as shown in proposition 2.2. In other words, this Gaussian constant volatility model is simple enough to retain key features of the finite dimensional setting.

\(^{11}\) See, for example, Mehra and Prescott (1985), Rietz (1988), Weil (1989), Kocherlakota (1990), Alvarez and Jermann (2001), Cogley and Sargent (2008), or Collin-Dufresne et al. (2016).
of the form \( Q_t = q(X_t) \) where \( q \) satisfies

\[
q(x) = \mathbb{E} [\Phi_{t+1} (Q_{t+1} + 1) \mid X_t = x]
\]

for each \( x \in X \). Using (20a)–(20b) and the formula for the lognormal expectation, this can be written as

\[
q(x) = \beta \exp \left[ \mu_d - \gamma \mu_c + (1 - \gamma)x + \frac{\sigma_d^2 + (\gamma \sigma_c)^2}{2} \right] \sum_{y \in X} (q(y) + 1) \Pi(x, y),
\]

or, stated as a vector equation, \( q = V(q + \mathbb{1}) \). Here \( \mathbb{1} \) is a column vector of ones of size \( |X| \) and \( V \) is the valuation matrix, with \((x,y)\)-th element

\[
V(x,y) := \beta \exp \left[ \mu_d - \gamma \mu_c + (1 - \gamma)x + \frac{\sigma_d^2 + (\gamma \sigma_c)^2}{2} \right] \Pi(x, y).
\] (26)

As is well understood, the Neumann Series Theorem implies that a solution to \( q = V(q + \mathbb{1}) \) exists whenever \( r(V) < 1 \), where \( r(V) \) is the spectral radius of \( V \). In view of (14) and proposition 2.2, \( r(V) < 1 \) is the same statement as \( \mathcal{L}_\Phi < 0 \) whenever \( X \) is finite. Thus, in the finite state setting, the condition \( \mathcal{L}_\Phi < 0 \) from theorem 2.1 is identical to the standard condition \( r(V) < 1 \). This means that the standard condition \( r(V) < 1 \) is not just sufficient for existence and uniqueness of an equilibrium price-dividend process, as had previously been understood, but also necessary whenever \( \{X_t\} \) is irreducible.\(^{12}\)

3.3. Habit Persistence. There is a large literature on asset prices in the presence of consumption externalities and habit formation (see, e.g., Abel (1990) and Campbell and Cochrane (1999)). In the “external” habit formation setting of Abel (1990) and Calin et al. (2005), the growth adjusted SDF takes the form

\[
M_{t+1} \frac{D_{t+1}}{D_t} = k_0 \exp((1 - \gamma)(\rho - \alpha)X_t)
\] (27)

where \( k_0 := \beta \exp(b(1 - \gamma) + \sigma^2(\gamma - 1)^2/2) \) and \( \alpha \) is a preference parameter. The state sequence \( \{X_t\} \) obeys

\[
X_{t+1} = \rho X_t + b + \sigma \eta_{t+1} \quad \text{with} \quad -1 < \rho < 1 \quad \text{and} \quad \{\eta_t\} \overset{iid}{\sim} N(0,1).
\] (28)

\(^{12}\)The logic is as follows: theorem 2.1 states that \( \mathcal{L}_\Phi < 0 \) is necessary whenever assumptions 2.1–2.3 hold. Since \( X \) is finite, assumption 2.3 holds automatically, by proposition 2.2. Assumption 2.1 is also true in the current setting. Hence necessity requires only assumption 2.1, which is irreducibility.
The parameter $b$ is equal to $x_0 + \sigma^2(1 - \gamma)$ where $x_0$ represents mean constant growth rate of the dividend of the asset.

The price-dividend ratio associated with this stochastic discount factor satisfies the forward recursion (7) from example 2.1 and, by theorem 2.1, there exists a unique price-dividend with finite second moment (we set $p = 2$ in theorem 2.1), if $L_\Phi^2 < 0$ and assumptions 2.1–2.3 are satisfied. Assumptions 2.1–2.3 can be verified when $p = 2$ in almost identical manner to the corresponding discussion in section 3.1. Hence, by theorem 2.1, a unique equilibrium price-dividend ratio with finite second moment exists if and only if $L_\Phi^2 < 0$.

An analytical expression for $L_\Phi^2$ can be obtained using similar techniques to those employed in section 3.1. Stepping through the algebra shows that

$$L_\Phi^2 = \ln k_0 + (1 - \gamma) (\rho - \alpha) \frac{b}{1 - \rho} + \frac{(1 - \gamma)^2 (\rho - \alpha)^2}{2} \frac{\sigma^2}{(1 - \rho)^2}.$$  \hspace{1cm} (29)

A unique equilibrium price-dividend ratio exists in $\mathcal{H}_2$ if and only this term is negative. The intuition behind the expression (29) is analogous to (25) in section 3.1.

To give some basis for comparison, let us contrast the condition $L_\Phi^2 < 0$ with the sufficient condition for existence and uniqueness of an equilibrium price-dividend ratio found in proposition 1 of Calin et al. (2005), which implies a one step contraction. Their test is of the form $\tau < 1$, where $\tau$ depends on the parameters of the model (see equation (7) of Calin et al. (2005) for details). Since the condition $L_\Phi^2 < 0$ requires only eventual contraction, rather than one step contraction, we can expect it to be significantly weaker than the condition of Calin et al. (2005).

Figure 1 supports this conjecture. The left sub-figure shows $\ln \tau$ at a range of parameterizations. The right sub-figure shows $L_\Phi^2$ at the same parameters, evaluated using (29). The horizontal and vertical axes show grid points for the parameters $\beta$ and $\sigma$ respectively. For both sub-figures, $(\beta, \sigma)$ pairs with test values strictly less than zero (points to the south west of the 0.0 contour line) are where the respective condition holds. Points to the north west of this contour line are where it fails.\(^\text{13}\)

Inspection of the figure shows that the sufficient condition in Calin et al. (2005) fails for many parameterizations that do in fact have unique stationary Markov equilibria. That is, there are many empirically relevant $(\beta, \sigma)$ pairs such that $L_\Phi^2 < \hspace{1cm} \text{(29)}$

\(^\text{13}\)The parameters held fixed in figure 1 are $\rho = -0.14$, $\gamma = 2.5$, $x_0 = 0.05$ and $\alpha = 1$.\)
0, indicating existence and uniqueness of a solution with finite second moment, and yet \( \ln \tau > 0 \). Note also that, because \( \mathcal{L}^2_\Phi < 0 \) is both necessary and sufficient in our setting, the 0.0 contour line in the right sub-figure is an exact delineation between stable and unstable parameterizations.

3.4. Long-Run Risk With Stochastic Volatility. Next we turn to an asset pricing model with Epstein–Zin utility and stochastic volatility in cash flow and consumption estimated by Bansal and Yaron (2004). Preferences are represented by the continuation value recursion

\[
V_t = \left[ (1 - \beta) C_t^{1 - 1/\psi} + \beta \{ \mathcal{R}_t (V_{t+1}) \}^{1 - 1/\psi} \right]^{1/(1 - 1/\psi)},
\]

where \( \{C_t\} \) is the consumption path and \( \mathcal{R}_t \) is the certainty equivalent operator

\[
\mathcal{R}_t(Y) := (E_t Y^{1 - \gamma})^{1/(1 - \gamma)}.
\]

The parameter \( \beta \in (0, 1) \) is a time discount factor, \( \gamma \) governs risk aversion and \( \psi \) is the elasticity of intertemporal substitution. Dividends and consumption grow according to

\[
\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_t \eta_{c,t+1}, \tag{32a}
\]

\[
\ln(D_{t+1}/D_t) = \mu_d + \alpha z_t + \phi_d \sigma_t \eta_{d,t+1}, \tag{32b}
\]

\[
z_{t+1} = \rho z_t + \phi_z \sigma_t \eta_{z,t+1}, \tag{32c}
\]

\[
\sigma_{t+1}^2 = \max \left\{ \nu \sigma_t^2 + d + \phi \sigma \eta_{\sigma,t+1}, 0 \right\}. \tag{32d}
\]
Here \( \{\eta_{i,t}\} \) are IID and standard normal for \( i \in \{d, c, z, \sigma\} \). The state \( X_t \) can be represented as \( X_t = (z_t, \sigma_t) \). The (growth adjusted) SDF process associated with this model is

\[
\Phi_{t+1} := M_{t+1} \frac{D_{t+1}}{D_t} = \beta^\theta \frac{D_{t+1}}{D_t} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( \frac{W_{t+1}}{W_t - 1} \right)^{\theta - 1},
\]

where \( W_t \) is the aggregate wealth-consumption ratio and \( \theta := (1 - \gamma) / (1 - 1/\psi) \).\(^{14}\)

To obtain the aggregate wealth-consumption ratio \( \{W_t\} \) we exploit the fact that \( W_t = w(X_t) \) where \( w \) solves the Euler equation

\[
\beta^\theta \mathbb{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left( \frac{w(X_{t+1})}{w(X_t) - 1} \right)^\theta \right] = 1.
\]

Rearranging and using the expression for consumption growth given above, this equality can be expressed as

\[
w(z, \sigma) = 1 + [K w^\theta(z, \sigma)]^{1/\theta},
\]

where \( K \) is the operator

\[
K g(z, \sigma) = \beta^\theta \exp \left\{ (1 - \gamma)(\mu_c + z) + \frac{(1 - \gamma)^2 \sigma^2}{2} \right\} \Pi g(z, \sigma)
\]

In this expression, \( \Pi g(z, \sigma) \) is the expectation of \( g(z_{t+1}, \sigma_{t+1}) \) given the state’s law of motion, conditional on \( (z_t, \sigma_t) = (z, \sigma) \).

The existence of a unique solution \( w = w^* \) to (3.4) in \( \mathcal{H}_1 \) under the parameterization used in Bansal and Yaron (2004) is established in Borovička and Stachurski (2017) when the innovation terms \( \{\eta_{i,t}\} \) are truncated, so that the state space is a compact subset of \( \mathbb{R}^2 \). In what follows, we compute \( w^* \) using the iterative method described in Borovička and Stachurski (2017) and recover \( W_t \) as \( w^*(X_t) \) for each \( t \).

As discussed in detail in appendix A, to approximate the stability exponent \( L_\Phi \), we can use Monte Carlo, generating independent paths for the SDF process \( \{\Phi_t\} \) and averaging over them to estimate the expectation on the right hand side of (12).

\(^{14}\)For a derivation see, for example, Bansal and Yaron (2004), p. 1503.
In computing the product $\prod_{t=1}^{n} \Phi_t$ we used (32) and (33) to express it as

$$\prod_{t=1}^{n} \Phi_t = (\beta^\theta \exp(\mu_d - \gamma \mu_c))^n \times \exp \left( (\alpha - \gamma) \sum_{t=1}^{n} z_t - \gamma \sum_{t=1}^{n} \sigma_{t, c, t+1} + \varphi_d \sum_{t=1}^{n} \sigma_{t, d, t+1} + (\theta - 1) \sum_{t=1}^{n} \hat{w}_t \right), \quad (35)$$

where $\hat{w}_{t+1} = \ln[W_{t+1}/(W_t - 1)]$.

At the parameter values using in Bansal and Yaron (2004) and based on the Monte Carlo method discussed above, we estimate that $L_\Phi = -0.00388$ implying the existence of a unique equilibrium price-dividend ratio function in $H_1$.\(^{15}\)

While this value is close to zero, we find that significant shifts in parameters are required to cross the contour $L_\Phi = 0$.

For example, figure 2 shows $L_\Phi$ calculated at a range of parameter values in the neighborhood of the Bansal and Yaron (2004) specification via a contour map. The parameter $\alpha$ is varied on the horizontal axis, while $\mu_d$ is on the vertical axis. Other parameters are held fixed at the Bansal and Yaron (2004) values. The black contour line shows the boundary between stability and instability. Not surprisingly, the test value increases with the cash flow growth rate $\mu_d$. In this region of the parameter space, it also declines with $\alpha$, because an increase in $\alpha$ with $\gamma > \alpha$ reduces the covariance between cash flow growth and discounting captured by the term $(\alpha - \gamma) \sum_{t=1}^{n} z_t$ in (35). However, we can see that $L_\Phi < 0$ fails only after significant deviations of $\alpha$ and $\mu_d$ from their estimated values.

3.5. Long-Run Risk Part II. Now we repeat the analysis in section 3.4 but using instead the dynamics for consumption and dividends in Schorfheide et al. (2018),

\(^{15}\)The value shown is the mean of 1,000 Monte Carlo draws $L_\Phi(n,m)$, where the latter is defined in (39) of appendix A. For each draw, $n$ and $m$ in in this calculation were set to 1,000 and 10,000 respectively. The standard deviation was less than 0.001. Following Bansal and Yaron (2004), the parameters used were $\gamma = 10.0, \beta = 0.998, \psi = 1.5, \mu_c = 0.0015, \rho = 0.979, \varphi_z = 0.044, \nu = 0.987, d = 7.9092e-7, \varphi_d = 2.3e-6, \mu_d = 0.0015, \alpha = 3.0$ and $\varphi_d = 4.5$. See table IV on page 1489.
Figure 2. The exponent $L_\Phi$ for the Bansal–Yaron model

which are given by

$$\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_{c,t} \eta_{c,t+1},$$
$$\ln(D_{t+1}/D_t) = \mu_d + \alpha z_t + \delta \sigma_{c,t} \eta_{c,t+1} + \sigma_{d,t} \eta_{d,t+1},$$
$$z_{t+1} = \rho z_t + (1 - \rho^2)^{1/2} \sigma_{z,t} \nu_{t+1},$$
$$\sigma_{i,t} = \phi_i \bar{\sigma} \exp(h_{i,t}),$$
$$h_{i,t+1} = \rho_i h_i + \sigma_i \tilde{\zeta}_{i,t+1}, \quad i \in \{c, d\}.$$  

The innovation vectors $\eta_t = (\eta_{c,t}, \eta_{d,t})$ and $\xi_t := (\nu_t, \xi_{z,t}, \xi_{c,t}, \xi_{d,t})$ are IID over time, mutually independent and standard normal in $\mathbb{R}^2$ and $\mathbb{R}^4$ respectively. The state can be represented as the four dimensional vector $X_t := (z_t, h_{z,t}, h_{c,t}, h_{d,t})$. Otherwise the analysis and methodology radius is similar to section 3.4. The product of the growth adjusted stochastic discount factors over $n$ period from $t = 1$ is

$$\prod_{t=1}^{n} \Phi_t = (\beta^\theta \exp(\mu_d - \gamma \mu_c))^n$$
$$\exp\left((\alpha - \gamma) \sum_{t=1}^{n} z_t + (\delta - \gamma) \sum_{t=1}^{n} \sigma_{c,t} \eta_{c,t+1} + \sum_{t=1}^{n} \sigma_{d,t} \eta_{d,t+1} + (\theta - 1) \sum_{t=1}^{n} \hat{w}_t\right)$$
As in section 3.4, we generate this product many times and then average to obtain an approximation of $L_\Phi$. At the parameterization used in Schorfheide et al. (2018), this evaluates to $-0.001$, indicating the existence of a unique equilibrium price dividend ratio.\footnote{We used the posterior mean values from Schorfheide et al. (2018), setting $\beta = 0.999$, $\gamma = 8.89$, $\psi = 1.97$, $\mu_c = 0.0016$, $\rho = 0.987$, $\varphi_z = 0.215$, $\sigma = 0.0032$, $\phi_c = 1.0$, $\rho_{hz} = 0.992$, $\sigma_{hz} = \sqrt{0.0039}$, $\rho_{hc} = 0.991$, $\sigma_{hc} = \sqrt{0.0096}$, $\mu_d = 0.001$, $\alpha = 3.65$, $\delta = 1.47$, $\phi_d = 4.54$, $\rho_{hd} = 0.969$, and $\sigma_{hd} = \sqrt{0.0447}$. We set $n = 1,000$ and $m = 10,000$, and then drew 1,000 observations of the statistic $L_\Phi(n,m)$, as defined in (39) of appendix A. The mean of these 1,000 draws was $-0.00103$ and the standard deviation was 0.00080.}

Figure 3 shows the stability exponent $L_\Phi$ calculated at a range of parameter values in the neighborhood of the Schorfheide et al. (2018) specification. The parameter $\phi_d$ is varied on the horizontal axis, while $\mu_d$ is on the vertical axis. Other parameters are held fixed at the Schorfheide et al. (2018) values. The interpretation is analogous to that of figure 2 from section 3.4, as is the method of computation, with the dark contour line shows the exact boundary between stability and instability. Increases in both $\mu_d$ and $\varphi_d$ increase the long-run growth rate of the level of the discounted cash flow, and hence increase $L_\Phi$. As with figure 2, significant deviations in estimated parameter values are required to change the sign of $L_\Phi$.\footnote{We used the posterior mean values from Schorfheide et al. (2018), setting $\beta = 0.999$, $\gamma = 8.89$, $\psi = 1.97$, $\mu_c = 0.0016$, $\rho = 0.987$, $\varphi_z = 0.215$, $\sigma = 0.0032$, $\phi_c = 1.0$, $\rho_{hz} = 0.992$, $\sigma_{hz} = \sqrt{0.0039}$, $\rho_{hc} = 0.991$, $\sigma_{hc} = \sqrt{0.0096}$, $\mu_d = 0.001$, $\alpha = 3.65$, $\delta = 1.47$, $\phi_d = 4.54$, $\rho_{hd} = 0.969$, and $\sigma_{hd} = \sqrt{0.0447}$. We set $n = 1,000$ and $m = 10,000$, and then drew 1,000 observations of the statistic $L_\Phi(n,m)$, as defined in (39) of appendix A. The mean of these 1,000 draws was $-0.00103$ and the standard deviation was 0.00080.}
3.6. **Unbounded Utility and Stationary Dividends.** We can use theorem 2.1 to generalize a recent result of Brogueira and Schütze (2017). In their setting, \( \{X_t\} \) is a stationary Markov process with stochastic density kernel \( q(x, y) \), consumption satisfies \( C_t = c(X_t) \) for some measurable and positive function \( c \) and, in the forward looking equation (6), \( \Phi_t = \beta \) and \( G_t = u'(c(X_t))c(X_t) \), where \( \beta \in (0, 1) \) and \( u \) is a concave and strictly increasing (but not necessarily bounded) utility function on \( \mathbb{R}_+ \).

Consider the conditions of theorem 2.1 at \( p = 2 \). Since \( \mathcal{L}_\Phi^2 = \ln \beta < 0 \), a unique equilibrium price process with finite second moment exists whenever assumptions 2.1–2.3 hold. Assumptions 2.1 and 2.2 are obviously true and assumption 2.3 holds whenever \( M := \mathbb{E}[u'(c(X_t))c(X_t)]^2 \) is finite and \( Vh(x) = \beta \int h(y)q(x, y)\,dy \) is an eventually compact linear operator on \( L_2(X, \mathbb{R}, \pi) \). These conditions will hold if, as in Brogueira and Schütze (2017), we take \( X = \mathbb{R} \), utility is CRRA as in (19), \( c(x) \) is the exponential function \( a \exp(x) \) for some \( a > 0 \) and \( q(x, y) = N(\rho x, \sigma) \) for some \( \sigma > 0 \) and \( |\rho| < 1 \). Indeed, under this specification, we have \( M = \mathbb{E}\exp(2(1 - \gamma)X_t) < \infty \), since \( X_t \) is Gaussian. Eventual compactness of \( V \) is true by proposition B.1 in the appendix.

### 4. Extensions

The result on necessity and sufficiency of \( \mathcal{L}_\Phi^1 < 0 \) in section 2 was established in a stationary Markov environment satisfying a number of auxiliary assumptions. What if these conditions are dropped? In this section we study the implications of \( \mathcal{L}_\Phi^1 := \mathcal{L}_\Phi < 0 \) in a setting with far less structure.

#### 4.1. An Existence Result.

One way to obtain a solution to the forward looking model (6) is to iterate forward in time, which leads to, in the limit

\[
\bar{Y}_t := \mathbb{E}_t \sum_{n=1}^{\infty} \left( \prod_{i=1}^{n} \Phi_{t+i} \right) G_{t+n}.
\]

In asset pricing terms, when prices are well defined, current price equals current expectation of the total sum of lifetime cash flow, appropriately discounted. We now seek conditions required for the right hand side of (36) to be finite, and for the resulting stochastic process \( \{\bar{Y}_t\} \) to solve (6).

**Assumption 4.1.** There exists a constant \( m \) such that \( \mathbb{P}\{G_t \leq m\} = 1 \) for all \( t \geq 0 \).
Assumption 4.1 can potentially be weakened but it already includes many important cases. For example, assumption 4.1 obviously holds in the setting of example 2.1, which handles nonstationary cash flows with stationary growth paths.

**Proposition 4.1.** Let assumption 4.1 hold. If, for some \( t \geq 0 \), we have
\[
\limsup_{n \to \infty} \frac{1}{n} \ln \left\{ \mathbb{E} \prod_{i=1}^{n} \Phi_{t+i} \right\} < 0,
\]
then \( \bar{Y}_t \) defined in (36) is finite with probability one. If, in addition, (37) holds for all \( t \geq 0 \), then the stochastic process \( \{\bar{Y}_t\} \) solves (6).

Proposition 4.1 only provides a sufficient condition, rather than a necessary and sufficient one. But the conditions imposed on the primitives are far weaker than theorem 2.1, which required a stationary Markov structure and assumptions 2.1–2.3. Below is a simple but useful corollary to proposition 4.1, focused on the stationary case. It relates to the stability exponent of the SDF process, as defined in (12).

**Corollary 4.2.** Let assumption 4.1 hold. If \( \{\Phi_t\} \) is stationary and \( L_\Phi < 0 \), then the stochastic process \( \{\bar{Y}_t\} \) is finite with probability one and solves (6).

Corollary 4.2 is immediate from stationarity of \( \{\Phi_t\} \) and proposition 4.1.

### 4.2. Application: A Theorem of Lucas

Corollary 4.2 can be used to obtain a classic result of Lucas (1978) on existence of equilibrium asset prices under far weaker conditions. In the problem of Lucas (1978), the price process obeys (1), where \( P_t \) is the price of a claim to the aggregate endowment stream \( \{D_t\} \), and the stochastic discount factor is as given in (18). In equilibrium, \( C_t \) is equal to an endowment \( D_t \), which is itself a continuous function of a stationary Markov process. Following Lucas (1978) we take \( Y_t := P_t u'(C_t) \) to be the endogenous object rather than \( P_t \). After dividing the fundamental asset pricing equation (1) through by \( u'(C_t) \) and setting \( D_t = C_t \) for all \( t \), we obtain
\[
Y_t = \beta \mathbb{E}_t[Y_{t+1} + u'(C_{t+1})C_{t+1}].
\]

This is a version of (6) with \( \Phi_t = \beta \) and \( G_t = u'(C_t)C_t \). Assumption 4.1 holds because Lucas (1978) assumes that \( u \) is nonnegative, concave and bounded, which in turn gives \( G_t = u'(C_t)C_t \leq m \) for some finite constant \( m \). Finally, since \( \Phi_t = \beta \),
the left hand side of (37) is just \( \ln \beta \), which is strictly negative because \( \beta < 1 \). Hence corollary 4.2 applies and a \( \mathbb{P} \)-almost surely finite solution to (38) exists. Note that this argument does not use any part of the assumption that consumption is a fixed and continuous function of a stationary Markov process.

5. Conclusion

We developed a practical test for existence and uniqueness of equilibrium asset prices in infinite horizon arbitrage free settings. By eschewing the one-step contraction methods of earlier work and seeking instead restrictions that ensure contraction occurs “on average, eventually,” we find a test that is necessary as well as sufficient, and hence can provide exact delineation between stable and unstable models in realistic applications. Computational techniques are provided to ensure that the test can be implemented in complex quantitative applications.

In our applications, we focused on consumption-based asset pricing models. However, the theoretical results apply in the same way to other no-arbitrage settings where asset prices can be represented using recursion (1) with a positive marginal rate of substitution. Embedding this analysis into frameworks with endogenously determined consumption is left to future research.

Appendix A. Computing the Stability Exponent

The stability exponent \( L^p_\Phi \) plays a key role our results. In some cases it can be calculated analytically, as in (25) or (29). In others it needs to be computed. If the state space is small and finite, then, as shown in section 3.2, the exponent \( L^p_\Phi = L_\Phi \) is equal to the log of the spectral radius of a valuation matrix, as shown in (14), and can therefore be obtained by numerical linear algebra: First compute the eigenvalues of \( V \) and then take the maximum in terms of modulus to obtain \( r(V) \). Now set \( L_\Phi = \ln r(V) \).

For cases where the state space is infinite and no analytical expression for \( L_\Phi \) exists, we consider two methods, both of which have advantages in certain settings. The first is based on discretization and the second is a Monte Carlo method. The discretization method works well for low dimensional state processes but is susceptible to the curse of dimensionality. The Monte Carlo method is slower when
the state space is finite with a small number of elements but less susceptible to the curse of dimensionality—and also highly parallelizable.

A.1. **Discretization.** If \( \Phi_t \) is a function of a stationary Markov process but the state space for that Markov process is not finite, we can discretize it (see, e.g., Tauchen and Hussey (1991), Rouwenhorst (1995), or Farmer and Toda (2017)). Once that procedure has been carried out, the remaining steps are the same as for the finite state Markov case described above. In what follows we investigate this procedure, finding support for its efficiency when the state space is relatively small.

Our experiment is based on the constant volatility model from section 3.1, where the analytical expression for \( L^p_\Phi = L_\Phi \) exists. We discretize the Gaussian AR(1) state process (20c) using Rouwenhorst’s method, compute the valuation matrix \( V \) in (26) corresponding to this discretized state process, calculate the spectral radius \( r(V) \) using linear algebra routines and, from there, compute the associated value for the stability exponent via (14). Finally, we compare the result with the true value of \( L_\Phi \) obtained from the analytical expression (25).

Figure 4 shows this comparison when the utility parameter \( \gamma \) is set to 2.5 and the consumption and dividend parameters are set to the values in table I of Bansal and Yaron (2004).\(^{17}\) The vertical axis shows the value of \( L_\Phi \). The horizontal axis shows the level of discretization, indexed by the number of states for \( \{X_t\} \) generated at the Rouwenhorst step. The true value of \( L_\Phi \) at these parameters, as calculate from (25), is \( -0.0031545 \). The discrete approximation of \( L_\Phi \) is accurate up to six decimal places whenever the state space has more than 6 elements. Thus, the discrete approximation is sufficiently accurate to implement the test \( L_\Phi < 0 \) even for relatively coarse discretizations. Moreover, as shown in the figure, the approximation of \( L_\Phi \) converges to the true value as the number of states increases. We experimented with other parameter values and found similar results.

A.2. **A Monte Carlo Method.** One issue with the discretization based method just discussed is that the algorithm is computationally inefficient when the state space is large. For this reason we also propose a Monte Carlo method that requires only the ability to simulate the SDF process \( \{\Phi_t\} \). This method is less susceptible to the

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\(^{17}\)In particular, \( \mu_c = \mu_d = 0.0015, \rho = 0.979, \sigma = 0.0034, \sigma_c = 0.0078, \sigma_d = 0.035 \) and \( \varphi = 1.0 \).
Figure 4. Accuracy of discrete approximation of $\mathcal{L}_\Phi$.

The idea behind the Monte Carlo method is to approximate $\mathcal{L}_\Phi$ via

$$\mathcal{L}_\Phi(n, m) := \frac{1}{n} \ln \left\{ \frac{1}{m} \sum_{j=1}^{m} \prod_{i=1}^{n} \Phi_i^{(j)} \right\},$$

(39)

where each $\Phi_1^{(j)}, \ldots, \Phi_n^{(j)}$ is an independently simulated path of $\{\Phi_t\}$, and $n$ and $m$ are suitably chosen integers. The idea relies on the strong law of large numbers, which yields $\frac{1}{m} \sum_{j=1}^{m} \prod_{i=1}^{n} \Phi_i^{(j)} \to \mathbb{E} \prod_{i=1}^{n} \Phi_i$ with probability one, combined with the fact that $Z_n \to Z$ almost surely implies $g(Z_n) \to g(Z)$ almost surely whenever $g: \mathbb{R} \to \mathbb{R}$ is continuous. However, these are only asymptotic results and our concern here is sufficiently good performance in finite samples.

Table 1 analyzes this issue. Here we again use the constant volatility model from section 3.1, comparing Monte Carlo approximations of $\mathcal{L}_\Phi$ with the true value obtained via the analytical expression given in (25). The consumption and dividend growth parameters are again chosen to match table I of Bansal and Yaron (2004), as in footnote 17. The true value of $\mathcal{L}_\Phi$ calculated from the analytical expression (25) is $-0.0031545$, as shown in the caption for the table. The interpretation of $n$ and $m$ in the table is consistent with the left hand side of (39). For each $n, m$ pair, we compute $\mathcal{L}_\Phi(n, m)$ 1,000 times using independent draws and present the mean and the
TABLE 1. Monte Carlo spectral radius estimates when $\mathcal{L}_\Phi = -0.0031545$

<table>
<thead>
<tr>
<th></th>
<th>m = 1000</th>
<th>m = 2000</th>
<th>m = 3000</th>
<th>m = 4000</th>
<th>m = 5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 250</td>
<td>-0.0033183</td>
<td>-0.0032524</td>
<td>-0.0032434</td>
<td>-0.0032533</td>
<td>-0.0032353</td>
</tr>
<tr>
<td></td>
<td>(0.000099)</td>
<td>(0.000065)</td>
<td>(0.000056)</td>
<td>(0.000047)</td>
<td>(0.000042)</td>
</tr>
<tr>
<td>n = 500</td>
<td>-0.0032045</td>
<td>-0.0032149</td>
<td>-0.0031948</td>
<td>-0.0031907</td>
<td>-0.0031922</td>
</tr>
<tr>
<td></td>
<td>(0.000080)</td>
<td>(0.000058)</td>
<td>(0.000045)</td>
<td>(0.000040)</td>
<td>(0.000036)</td>
</tr>
<tr>
<td>n = 750</td>
<td>-0.0031985</td>
<td>-0.0031841</td>
<td>-0.0031748</td>
<td>-0.0031784</td>
<td>-0.0031890</td>
</tr>
<tr>
<td></td>
<td>(0.000080)</td>
<td>(0.000054)</td>
<td>(0.000044)</td>
<td>(0.000041)</td>
<td>(0.000038)</td>
</tr>
</tbody>
</table>

standard deviation of the sample in the corresponding cell. We find that the Monte Carlo approximation is accurate up to three decimal places in all simulations, and up to four decimal places when $n = 750$. Standard deviations are small. The table suggests that, at least for this model, the Monte Carlo method is precise enough to determine the sign of $\mathcal{L}_\Phi$.

APPENDIX B. PROOFS

If $\mathcal{E}$ is a Banach lattice, then an ideal in $\mathcal{E}$ is, as usual, a vector subspace $L$ of $\mathcal{E}$ with $x \in L$ whenever $|x| \leq |y|$ and $y \in L$. The spectral radius of a bounded linear operator $M$ from $\mathcal{E}$ to itself is the supremum of $|\lambda|$ for all $\lambda$ in the spectrum of $A$. The operator $M$ is called compact if the image under $M$ of the unit ball in $\mathcal{E}$ has compact closure. $M$ is called eventually compact if there exists an $i \in \mathbb{N}$ such that $M^i$ is compact. $M$ is called positive if it maps the positive cone of $\mathcal{E}$ into itself. A positive linear operator $M$ is called irreducible if the only closed ideals $J \subset \mathcal{E}$ satisfying $M(J) \subset J$ are $\{0\}$ and $\mathcal{E}$. See Abramovich et al. (2002) or Meyer-Nieberg (2012) for more details.

If $X$ is a Polish space, $\pi$ is a finite Borel measure on $X$ and $p \geq 1$, then $L_p(\pi) := L_1(X, \mathbb{R}, \pi)$ denotes the set of all Borel measurable functions $f$ from $X$ to $\mathbb{R}$ satisfying $\int |f|^p \, d\pi < \infty$. The norm on $L_p(\pi)$ is $\|f\| := \left( \int |f|^p \, d\pi \right)^{1/p}$. Functions equal $\pi$-almost everywhere are identified. Convergence on $L_p(\pi)$ is with respect to the norm topology generated by $\| \cdot \|$. We write $f \leq g$ for $f, g$ in $L_p(\pi)$ if $f \leq g$ holds pointwise $\pi$-almost everywhere. We write $f \ll g$ if $f < g$ holds pointwise $\pi$-almost everywhere. The positive cone of $L_p(\pi)$ is all $f \in L_p(\pi)$ with $f \geq 0$. 

We denote this set by $H_P$, which conforms with our previous definition (see, in particular, theorem 2.1).

B.1. **Operator Compactness in Spaces of Summable Functions.** Assumption 2.3 requires that, for some $p \geq 1$, the operator $V$ is eventually compact as a linear map from $L_p(X, \mathbb{R}, \pi)$ to itself. In this section we discuss some sufficient conditions and state a result that was used in section 3.\(^{18}\)

One sufficient condition is as follows: $V$ will be eventually compact if there is a bounded linear operator $M$ such that $V \leq M$ pointwise on $L_p(\pi)$ and $M$ is eventually compact. Indeed, in that case there exists a $k \in \mathbb{N}$ such that $M^k$ is compact and, since each $L_p$ space has order continuous norm (Meyer-Nieberg (2012), §2.4), it follows from corollary 2.37 in Abramovich et al. (2002) that $V^{2k}$ is compact. Hence $V$ is eventually compact.

Next we give a sufficient condition focused on the applications in section 3. Take $X = \mathbb{R}$ and $p = 2$. In the proposition below, $q$ is a stochastic density kernel on $\mathbb{R}^2$ with stationary density $\pi$ and two step density kernel $q^2$. The statement that $q$ is time-reversible means that $q(x, y)\pi(x) = q(y, x)\pi(y)$ for all $x, y \in \mathbb{R}$. (A number of our results use the fact that $q(x, \cdot) = N(\rho x, \sigma^2)$ for some $\sigma > 0$ and $|\rho| < 1$ implies that $q$ is time-reversible. See, e.g., O’Donnell (2014).)

**Proposition B.1.** Let $M$ be an operator that maps $f$ in $L_2(\pi)$ into

$$Mf(x) = g(x) \int f(y)q(x, y) \, dy \quad (x \in \mathbb{R}),$$

where $g$ is a measurable function from $\mathbb{R}$ to $\mathbb{R}_+$. If $q$ is time-reversible and

$$\int g(x)q^2(x, x) \, dx < \infty,$$

then $M$ is a compact linear operator on $L_2(\pi)$.

**Proof.** We can express the operator $M$ as

$$Mf(x) = \int f(y)k(x, y)\pi(y) \, dy \quad \text{where} \quad k(x, y) := \frac{g(x)q(x, y)}{\pi(y)}.$$

---

\(^{18}\)It is worth nothing that, since $V$ is a positive operator and obviously linear, $V$ is a bounded linear operator on $L_p(\pi)$ whenever it maps $L_p(\pi)$ to itself (see Abramovich et al. (2002), theorem 1.31). In particular, boundedness of $V$ need not be separately checked.
By theorem 6.11 of Weidmann (2012), the operator $M$ will be Hilbert–Schmidt in $L_2(\pi)$, and hence compact, if the kernel $k$ satisfies
\[ \int \int k(x, y)^2 \pi(x) \pi(y) \, dx \, dy < \infty. \]

Using the definition of $k$ and the time-reversibility of $q$, this translates to
\[ \int g(x) \int q(x, y)q(y, x) \, dy \, dx < \infty. \]

This completes the proof because, by definition, $q^2(x, x) = \int q(x, y)q(y, x) \, dy$. □

B.2. Remaining Proofs. Let $X$ be the state space, as in section 2.1. Throughout all of the following we take assumptions 2.1, 2.2 and 2.3 to be in force. The symbol $p$ represents the constant in assumption 2.3. The positive cone of $L_p(\pi)$ is all $f \in L_p(\pi)$ with $f \geq 0$. We denote this set by $H_p$, which conforms with our previous definition (see, in particular, theorem 2.1).

As before, $\Pi$ is a stochastic kernel on $X$ and $\{X_t\}$ is a stationary Markov process on $X$ with stochastic kernel $\Pi$ and common marginal distribution $\pi$.\(^{19}\) Let $\Pi^n$ denote the $n$-step stochastic kernel corresponding to $\Pi$. The symbol $E_x$ will indicate conditioning on the event $X_0 = x$, so that, for any $h \in L_1(\pi)$ and any $n \in \mathbb{N}$, we have
\[ E_x h(X_n) = \int h(x) \Pi^n(x, dy). \quad (42) \]

**Lemma B.2.** For any $h \in H_p$ and all $x \in X$ we have
\[ V^n h(x) = E_x \prod_{i=1}^n \Phi_i h(X_n). \quad (43) \]

**Proof.** Equation (43) holds when $n = 1$ because
\[ V h(x) = \int \int \phi(x, y, \eta) v(\eta) h(y) \Pi(x, dy) \, dy = E_x \Phi_1 h(X_1). \]

Now suppose (43) holds at arbitrary $n \in \mathbb{N}$. We claim it also holds at $n + 1$. Indeed,
\[ V^{n+1} h(x) = E_x \Phi_1 V^n h(X_1) = E_x \Phi_1 \prod_{i=2}^{n+1} \Phi_i h(X_{n+1}) = E_x \prod_{i=1}^{n+1} \Phi_i h(X_{n+1}). \]

An application of the law of iterated expectations completes the proof. □

\(^{19}\) In other words, $\Pi$ is a function from $(X, \mathcal{B})$ to $[0, 1]$ such that $B \mapsto \Pi(x, B)$ is a probability measure on $(X, \mathcal{B})$ for each $x \in X$, and $x \mapsto \Pi(x, B)$ is $\mathcal{B}$-measurable for each $B \in \mathcal{B}$. The process $\{X_t\}$ satisfies $\mathbb{P}\{X_{t+1} \in B \mid X_t = x\} = \Pi(x, B)$ for all $x$ in $X$ and $B \in \mathcal{B}$. 
Lemma B.3. For each $h \in \mathcal{H}_p$, $x \in X$ and $n \in \mathbb{N}$ we have

$$V^n h(x) = 0 \implies \int h(y) \Pi^n(x, dy) = 0.$$ 

Proof. Fix $h \in \mathcal{H}_p$, $x \in X$ and $n \in \mathbb{N}$, and suppose that $V^n h(x) = 0$. It follows from lemma B.2 that $E_x \prod_{i=1}^n \Phi_i h(X_n) = 0$, which in turn implies that $\prod_{i=1}^n \Phi_i h(X_n) = 0$ holds $\mathbb{P}_x$-a.s. But then $h(X_n) = 0$ holds $\mathbb{P}_x$-a.s., and hence $E_x h(X_n) = 0$. By (42), this is equivalent to $\int h(y) \Pi^n(x, dy) = 0$. □

Lemma B.4. For given $h \in \mathcal{H}_p$, the following statements are true:

(a) If $h \gg 0$, then $V^n h \gg 0$ for all $n \in \mathbb{N}$.

(b) If $h \neq 0$, then there exists an $n$ in $\mathbb{N}$ such that $V^n h \gg 0$.

Proof. Regarding part (a), it suffices to show this is true when $n = 1$, after which we can iterate. To this end, fix $h \in \mathcal{H}_p$ with $h > 0$ on $B \in \mathcal{B}$ with $\pi(B) = 1$. Suppose that

$$Vh(x) = \int h(y) \left( \int \phi(x, y, \eta) \nu(d\eta) \right) \Pi(x, dy) = 0.$$ 

Since $\phi$ is positive, we must then have $\Pi(x, B) = 0$. But $\pi$ is invariant, so $\pi(B) = \int \Pi(x, B) \pi(dx) = 0$. Contradiction. □

Lemma B.5. The valuation operator $V$ is irreducible on $L_p(\pi)$.

Proof. Suppose to the contrary that there exists a closed ideal $J$ in $L_p(\pi)$ such that $V$ is invariant on $J$ and $J$ is neither $\emptyset$ nor $L_p(\pi)$ itself. Since $J$ is a closed ideal in $L_p(\pi)$, there exists a set $B \in \mathcal{B}$ such that $J = \{ f \in L_p(\pi) : f = 0$ $\pi$-a.e. on $B \}$. Moreover, since $J$ is neither empty nor the whole space, it must be that, for this set $B$ that defines $J$, we have $0 < \pi(B) < 1$.

Because $V$ is invariant on $J$, we have $V^n h \in J$ for all $h \in J$ and $n \in \mathbb{N}$. In particular, $V^n 1_B \in J$ for all $n \in \mathbb{N}$. This means that $V^n 1_B(x) = 0$ for $\pi$-almost all $x \in B$ and all $n$ in $\mathbb{N}$. Fixing an $x \in B$ and applying lemma B.3, we then have $\Pi^n(x, B^c) = 0$ for all $n \in \mathbb{N}$. But $\pi(B) < 1$, so $\pi(B^c) > 0$. This contradicts irreducibility of the stochastic kernel $\Pi$ (see footnote 6 for the definition of the latter), which in turn violates assumption 2.2. □

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20See, for example, Gerlach and Nittka (2012), p. 765.
The following is a local spectral radius result suitable for \( L_p(\pi) \) that draws on results from Zabreiko et al. (1967) and Krasnosel’skii et al. (2012). (Suitability for \( L_p(\pi) \) is due to the fact that the interior of the positive cone can be empty.) The proof provided here is due to Miroslawa Zima (private communication). In the statement of the theorem, a quasi-interior element of the positive cone of a Banach lattice \( E \) is a nonnegative element \( h \) satisfying \( \langle h, g \rangle > 0 \) for any nonzero element of the positive cone of the dual space \( E^* \). (See Krasnosel’skii et al. (2012) for more details.)

**Theorem B.6.** Let \( h \) be an element of a Banach lattice \( E \) and let \( M \) be a positive and compact linear operator. If \( h \) is quasi-interior, then \( \| M^n h \|^{1/n} \to r(M) \) as \( n \to \infty \).

**Proof.** Let \( h \) and \( M \) be as in the statement of the theorem and let \( E_+ \) be the positive cone of \( E \). Let \( r(h, M) := \limsup_{n \to \infty} \| M^n h \|^{1/n} \). From the definition of \( r(M) \) it is clear that \( r(h, M) \leq r(M) \). To see that the reverse inequality holds, let \( \lambda \) be a constant satisfying \( \lambda > r(h, M) \) and let

\[
 h_\lambda := \sum_{n=0}^{\infty} \frac{M^n h}{\lambda^{n+1}}. \tag{44}
\]

The point \( h_\lambda \) is a well-defined element of \( E_+ \) by \( \limsup_{n \to \infty} \| M^n h \|^{1/n} < \lambda \) and Cauchy’s root test for convergence. It is also quasi-interior, since the sum in (44) includes the quasi-interior element \( h \), and since \( M \) maps \( E_+ \) into itself. Moreover, by standard Neumann series theory (e.g., Krasnosel’skii et al. (2012), theorem 5.1), the point \( h_\lambda \) also has the representation \( h_\lambda = (\lambda I - M)^{-1} h \), from which we obtain \( \lambda h_\lambda - M h_\lambda = h \). Because \( h \in E_+ \), this implies that \( M h_\lambda \leq \lambda h_\lambda \). Applying this last inequality, compactness of \( M \), quasi-interiority of \( h_\lambda \) and theorem 5.5 (a) of Krasnosel’skii et al. (2012), we must have \( r(M) \leq \lambda \). Since this inequality was established for an arbitrary \( \lambda \) satisfying \( \lambda > r(h, M) \), we conclude that \( r(h, M) \geq r(M) \).

We have now established that \( r(h, M) = r(M) \). Since \( M \) is compact, corollary 1 of Daneš (1987) gives

\[
 \limsup_{n \to \infty} \| M^n h \|^{1/n} = \lim_{n \to \infty} \| M^n h \|^{1/n}.
\]

The proof is now complete. \( \square \)

**Theorem B.7.** The growth exponent \( L_{\Phi} \) satisfies \( \exp(L_{\Phi}) = r(V) \).
Proof. Let $1 = 1_X$ be the function equal to unity everywhere on $X$. In view of lemma B.2, to prove theorem B.7 it suffices to show that

$$\lim_{n \to \infty} \|V^n 1\|^{1/n} = r(V).$$  \hspace{1cm} (45)

By assumption 2.3 we can choose an $i \in \mathbb{N}$ such that $V^i$ is a compact linear operator on $L_p(\pi)$. Fix $j \in \mathbb{N}$ with $0 \leq j \leq i - 1$. By lemma B.4 we know that $V^j 1$ is positive $\pi$-almost everywhere on $X$, and is therefore quasi-interior. \hspace{1cm} 21

As a result, theorem B.6 applied to $V^i$ with initial condition $h := V^i 1$ yields

$$\|V^{in+j} 1\|^{1/n} \to r(V^i) \hspace{1cm} (n \to \infty).$$

But $r(V^i) = r(V)^i$, so $\|V^{in+j} 1\|^{1/(in)} \to r(V)$ as $n \to \infty$. It follows that

$$\|V^{in+j} 1\|^{1/(in+j)} \to r(V).$$

As this is shown to be true for any integer $j$ satisfying $0 \leq j \leq i - 1$, we can conclude that (45) is valid. \hfill \Box

To prove theorem 2.1, we will also need the following two lemmas:

Lemma B.8. $T$ has a fixed point in $H_p$ if and only if there exist elements $g, h$ in $H_p$ such that $T^n g \to h$ as $n \to \infty$.

Proof. Suppose first that there exist $g, h$ in $H_p$ such that $T^n g \to h$ as $n \to \infty$. Since $T f = Vf + g$ and $V$ is a bounded linear operator on $L_p(\pi)$, we know that $T$ is continuous as a self-map on $L_p(\pi)$. Letting $g_n = T^n g$, we have $g_n \to h$ and hence, by continuity, $T g_n \to Th$. But, by the definition of the sequence $\{g_n\}$, we must also have $T g_n \to h$. Hence $Th = h$.

Conversely, if $T$ has a fixed point $f \in H_p$, then the condition in the statement of lemma B.8 is satisfied with $g = h = f$. \hfill \Box

Proposition B.9. If $T$ has a fixed point in $H_p$, then $L^p_\Phi < 0$.

21By the Riesz Representation Theorem, the dual space of $L_p(\pi)$ is isometrically isomorphic to $L_q(\pi)$ where $1/p + 1/q = 1$. If $g$ is a nonnegative and nonzero element of $L_q(\pi)$ then it is positive on a set of positive $\pi$ measure. Since $f \gg 0$ on $X$, the product $fg$ must be positive on a set of positive $\pi$ measure. Hence $\int fg \, d\pi > 0$, so $f$ is quasi-interior.
Proof. Let $V^*$ be the adjoint operator associated with $V$. Since $V$ is irreducible (see lemma B.5) and $V^i$ is compact for some $i$, the version of the Krein–Rutman theorem presented in lemma 4.2.11 of Meyer-Nieberg (2012) together with the Riesz Representation Theorem imply existence of an $e^*$ in the dual space $L_q(\pi)$ such that

$$e^* \gg 0 \quad \text{and} \quad V^*e^* = r(V)e^*. \quad (46)$$

Let $h$ be a fixed point of $T$ in $H_p$. Clearly $h$ is nonzero, since $T0 = V0 + \hat{g} = \hat{g}$ and $\hat{g}$ is not the zero function (see assumption 2.1). Moreover, since $h$ is a fixed point, we have $h = Vh + \hat{g}$ and hence, with the inner production notation $\langle \phi, f \rangle := \int \phi f \, d\pi$,

$$\langle e^*, h \rangle = \langle e^*, Vh \rangle + \langle e^*, \hat{g} \rangle = \langle V^*e^*, h \rangle + \langle e^*, \hat{g} \rangle = r(V)\langle e^*, h \rangle + \langle e^*, \hat{g} \rangle.$$ 

In other words,

$$(1 - r(V))\langle e^*, h \rangle = \langle e^*, \hat{g} \rangle.$$ 

Both $h$ and $\hat{g}$ are nonzero in $L_p(\pi)$ and $e^*$ is positive $\pi$-a.e., so $\langle e^*, h \rangle > 0$ and $\langle e^*, \hat{g} \rangle > 0$. It follows that $r(V) < 1$. By theorem B.7, we have $L_p^\Phi = \ln r(V)$, which proves the claim in the lemma. \hfill\□

Proof of theorem 2.1. By lemma B.8, (b) and (d) of theorem 2.1 are equivalent, so it suffices to show that (e) $\implies$ (c) $\implies$ (b) $\implies$ (a) $\implies$ (e). Of these, the implications (e) $\implies$ (c) $\implies$ (b) are trivial, and the fact that (b) $\implies$ (a) was established in proposition B.9. Hence we need only show that (a) $\implies$ (e).

To see that (a) implies (e), suppose that $L_p^\Phi < 0$. Then, by theorem B.7, we have $r(V) < 1$. Using Gelfand’s formula for the spectral radius, which states that $r(V) = \lim_{n \to \infty} \|V^n\|^{1/n}$ with $\|\cdot\|$ as the operator norm, we can choose $n \in \mathbb{N}$ such that $\|V^n\| < 1$. Then, for any $h, h' \in H_p$ we have

$$\|T^n h - T^n h'\| = \|V^n h - V^n h'\| = \|V^n (h - h')\| \leq \|V^n\| \cdot \|h - h'\|.$$ 

Observe that $H_p$ is closed in $L_p(\pi)$, since $L_p(\pi)$ is a Banach lattice. Hence $H_p$ is complete in the norm topology. Existence, uniqueness and global stability now follow from a well-known extension to the Banach contraction mapping theorem (see, e.g., p. 272 of Wagner (1982)).

Lastly, to see that (13) holds, suppose that (a)–(e) are true. Then $r(V) < 1$, which implies that $(I - V)^{-1}$ is well-defined on $H_p$ and equals $\sum_{i=0}^\infty V^i$ (see, e.g., theorem 2.3.1 and corollary 2.3.3 of Atkinson and Han (2009)). In particular, the fixed
point of $T$ is given by $h^* = \sum_{n=0}^{\infty} V^n \hat{g}$. Applying (43) to this sum verifies the claim in (13).

Proof of proposition 2.2. Fix $p \geq 1$. If assumptions 2.1–2.2 hold and $X$ is a finite set endowed with the discrete topology, then all functions from $X$ to $\mathbb{R}$ are measurable and have finite $p$-th moment, so $L_p(X, \mathbb{R}, \pi) = \mathbb{R}^X$ and $\mathcal{H}_p = \mathbb{R}^X$. It follows that $\hat{g} \in \mathcal{H}_p$ and $V$ is a bounded linear operator from $L_p(X, \mathbb{R}, \pi)$ to itself (since every linear operator mapping a finite dimensional normed vector space to itself is bounded). By the Heine–Borel theorem, bounded subsets in finite dimensional space have compact closure, so $V$ is also (eventually) compact. Thus, assumption 2.3 holds. Finally, $L^p_{\Phi} = L^1_{\Phi}$ by the identity in (14), since, in a finite dimensional normed linear space, the spectral radius is independent of the choice of norm (due to equivalence of norms combined with Gelfand’s formula for the spectral radius).

Proof of proposition 4.1. Fix $t \geq 0$. To prove that $\mathbb{P}\{\bar{Y}_t < \infty\} = 1$, it suffices to show that $\mathbb{E}\bar{Y}_t < \infty$, which, by the definition of $\bar{Y}_t$ and the law of iterated expectations, will hold whenever the infinite sum $\sum_{n=1}^{\infty} \mathbb{E} \prod_{i=1}^{n} \Phi_{t+i} G_{t+n}$ converges. (The expectation is passed through the sum by nonnegativity of the sum components combined with the Monotone Convergence Theorem, which is valid regardless of whether or not the sum is finite. See, e.g., Dudley (2002), theorem 4.3.2.) By Cauchy’s root criterion for convergence of sums, the sum converges whenever $A < 1$, where

$$A := \limsup_{n \to \infty} a_n^{1/n} \quad \text{with} \quad a_n := \mathbb{E} \prod_{i=1}^{n} \Phi_{t+i} G_{t+n}.$$

We have

$$\ln A \leq \limsup_{n \to \infty} \frac{1}{n} \ln \mathbb{E} \prod_{i=1}^{n} \Phi_{t+i} m = \limsup_{n \to \infty} \frac{1}{n} \ln \mathbb{E} \prod_{i=1}^{n} \Phi_{t+i} < 0,$$

where $m$ is the constant in assumption 4.1 and the final inequality is due to (37). Hence $A < 1$ and $\mathbb{E}\bar{Y}_t < \infty$, as was to be shown.

Now suppose that (37) holds for all $t \geq 0$. Then $\bar{Y}_t$ is finite with probability one for all $t$. Substituting the definition of $\bar{Y}_{t+1}$ into the right hand side of (6) and using the law of iterated expectations, it is straightforward to show that $\{\bar{Y}_t\}$ is a solution to (6). Details are omitted.
REFERENCES


454–476.


