

Risk Price Dynamics*

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Abstract

We present a novel approach to depicting asset pricing dynamics by characterizing shock exposures and prices for alternative investment horizons. We quantify the shock exposures in terms of elasticities that measure the impact of a current shock on future cash-flow growth. The elasticities are designed to accommodate nonlinearities in the stochastic evolution modeled as a Markov process. Stochastic growth in the underlying macroeconomy and stochastic discounting in the representation of asset values are central ingredients in our investigation. We provide elasticity calculations in a series of examples featuring consumption externalities, recursive utility, and jump risk.

Keywords: growth-rate risk, pricing, dynamics, elasticities, Markov process

JEL Classification: C52, E44, G12

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1 Introduction

We propose a new way to characterize risk price dynamics, and apply these methods to study several structural asset pricing models. In the methods of mathematical finance, risk prices are encoded using the familiar risk neutral transformation and the instantaneous risk-free rate. In structural models of macroeconomic risk, they are encoded in the stochastic discount factor process used to represent prices at alternative payoff horizons. Our aim is to reveal the pricing dynamics embedded in risk-neutral transformations or in stochastic discount factors by extending two types of methods: local risk prices and impulse response functions. Local risk prices give the reward expressed in terms of expected returns for alternative local exposures to risk, including shocks to the macroeconomy. Impulse response functions characterize how shocks today contribute to future values of a stochastic process such as macroeconomic growth or future cash flows. We develop related constructs, but ones that are tailored to the pricing of the exposure to macroeconomic risk. We achieve this by extending the concept of a local risk price by asking how the compensation for exposure to shocks changes as we alter the terminal or maturity date for the payoff. This leads us to construct shock-exposure and shock-price elasticities as functions of payoff horizons. Structural asset pricing models feature state dependence in risk premia as well as sensitivity to the payoff horizon. These risk premia depend on shock exposures and prices, and the elasticities we propose reflect both dependencies. Our methods show how state dependence alters these elasticities when the date of the shocks is shifted to time periods that are further in the future.

We believe that uncertainty about macroeconomic growth has important welfare implications and major consequences to market valuations of forward-looking assets. Exploring these phenomena requires the simultaneous study of stochastic growth and discounting, in contrast to the extensive literature on fixed income securities and the term structure of interest rates that abstracts from growth. Previous work¹ has sought to provide informative characterizations of risk premia for cash flows that grow stochastically over time and to extract the distinct contributions of risk exposure (the asset pricing counterpart to a quantity) and risk prices. We add to this literature by proposing and characterizing the state and investment horizon dependence of exposure and price elasticities.

While there have been quantitative and empirical successes through the use of *ad hoc* models of stochastic discount factors specified flexibly to enforce the absence of arbitrage, our aim is to reveal the pricing implications of structural models that allow us to truly answer the question “how does risk or uncertainty get priced?” The promise of such models

¹See, for instance, Lettau and Wachter (2007), Hansen and Scheinkman (2009a,b) and Hansen (2009).

is that they will allow researchers to assign values to the shocks identified in macroeconomic models and support welfare analyses that are linked to uncertainty. While reduced-form models continue to provide a convenient shortcut for presenting empirical evidence, we aim to provide a dynamic characterization of risk pricing that will support structural investigations that stretch models beyond the support of the existing data.

Many asset pricing models have state dependent movements in both means and volatilities. While Markovian, these models are fundamentally nonlinear. This makes their pricing implications over extended investment horizon more challenging to extract, but our methods aim to address this challenge. For example, the local pricing of the commonly used diffusion model exploits local normality to obtain simple characterizations. As we integrate over time this model becomes a more complicated “mixture of normals” model with nontrivial state dependence in the mixing. For typical state realizations, the thin tails of the normal density can be enlarged by this mixing across normal regimes in nontrivial ways. In our study of valuation through compounding stochastic growth and discounting, seemingly modest state dependence that is present over short investment intervals can be magnified over longer time intervals. While the elasticities we compute continue to exploit the local normality, we show how their impact can be magnified through this compounding. We also study models that include jump components to uncertainty.

1.1 Overview of the paper

Section 2.1 starts with the description of the economic environment with Brownian information structure. We specify a stationary Markov diffusion process for the underlying dynamics. This Markov process characterizes the increments of nonstationary functionals that capture growth and discounting. The paper provides a methodology for studying the impact of small perturbations of these functionals that are conveniently parameterized. We introduce these perturbations in Section 2.2 to construct the elasticities that interest us. These perturbations make marginal changes to the exposure of the multiplicative functional to alternative configurations of economic shocks. Economic motivation for the particular type of elasticities dictates how we construct these elasticities.

In this paper we construct alternative elasticities indexed by the investment horizon and the current Markov state. For a fixed investment horizon t and initial state x , we compute the response of the logarithm of the expected value of the perturbed multiplicative functional to marginal changes in the exposure. We call such an elasticity, scaled by the investment horizon, a *risk elasticity*. By localizing the change in the exposure to focus on the next instant, we build corresponding *shock elasticities*. Following Hansen and Scheinkman (2009b), we show that the shock elasticities are the building blocks for the risk elasticities.

A risk elasticity is a distorted expectation of an integral of shock elasticities over time. This distorted expectation is proposed and justified in Hansen and Scheinkman (2009a). The essential formula from this section is:

$$\text{risk elasticity} = \frac{1}{t} \frac{\hat{E} \left[\hat{e}(X_t) \int_0^t \varepsilon(X_u, t-u) du \mid X_0 = x \right]}{\hat{E} [\hat{e}(X_t) \mid X_0 = x]}$$

for investment horizon t where ε is the corresponding shock elasticity. The distorted expectation is captured by the \hat{E} expectation operator that is used along with the scaling by the random variable $\hat{e}(X_t)$. The construction of risk and shock elasticities is reported in Section 2 along with a characterization of the dependencies on the Markov state, the exposure date, and the length of the payoff horizon.

Risk premia depend on both the exposure of a cash flow to risk and the price of that exposure. The exposure plays the role of a quantity in standard demand theory. Given these two contributions, we are lead to compute two types of elasticities: an *exposure elasticity* and a *price elasticity*. Using these categories in conjunction with the ones mentioned in the previous paragraph, we construct the following four types of elasticities in Section 3:

- i) risk-price elasticity
- ii) shock-price elasticity
- iii) risk-exposure elasticity
- iv) shock-exposure elasticity

In Section 4, we provide a technical generalization of our analysis by using the Malliavin derivative from stochastic calculus.

We compute risk and shock elasticities with a series of examples in Sections 5, 6 and 7. Each of these sections can be read independently of the others. In Section 5 we display elasticities for a model with recursive utility preference in the spirit of Bansal and Yaron (2004) using a parameterization in Hansen et al. (2007). This model is a restricted version of a pricing model with affine dynamics in which both conditional means and conditional variances are linear in Markov states. The recursive utility model contributes a forward-looking component to the stochastic discount factor process represented using continuation values. We characterize the impact of this forward-looking component on price elasticities for alternative investment horizons. In Section 6 we contrast two specifications of models in which investors confront consumption externalities in their preferences, the so-called external habit models of Campbell and Cochrane (1999) and Santos and Veronesi (2008). These

models are known to induce nonlinearity in risk pricing. We document substantive differences in the shock and risk price elasticities across investment horizons.

We construct elasticities for models with discrete shifts in the conditional means and conditional volatilities in Section 7. These shifts are modeled as evolving according to a finite-state Markov chain specified in continuous time. In our computations with this specification, we use an estimated model of consumption dynamics from Bonomo and Garcia (1996) in conjunction with a recursive utility model of preferences. The recursive utility model is known to induce nonzero local prices of regime-shift risk. We extend this insight by studying the risk and shock price dynamics. For the three types of example economies, we use counterpart model economies in which investors have power utility preferences as benchmarks.

2 Markov pricing with Brownian information

We follow the construction in Hansen and Scheinkman (2009a,b) and Hansen (2009). Consider a Markov diffusion that solves:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

where W is a multivariate standard Brownian motion. In this model nonlinearity is captured by the specification of μ and σ . While the state variable X may well be stationary, we will use it as a building block for processes that grow or decay over time.

2.1 Growth and discounting

In econometric practice we often build models for the logarithms of processes. An example of such a model is

$$A_t = \int_0^t \beta(X_u)du + \int_0^t \alpha(X_u) \cdot dW_u.$$

We call the resulting process, denoted by A , an additive functional because it depends entirely on the underlying Markov process and it is constructed by integrating over the time scale. Nonlinearity may be present in the specification of β and α .

While it is convenient to take logarithms when building time series models, to represent values and prices it is necessary to study levels instead of logarithms. Thus to represent growth or decay, we use the exponential of an additive functional, $M_t = \exp(A_t)$. We will refer to M as a multiplicative functional parameterized by (β, α) . Ito's Lemma guarantees

that the local mean of M is

$$M_t \left[\beta(X_t) + \frac{|\alpha(X_t)|^2}{2} \right].$$

The multiplicative functional is a local martingale if its local mean is zero:

$$\beta(X_t) + \frac{|\alpha(X_t)|^2}{2} = 0.$$

There are two types of multiplicative functionals that we feature: we use one to represent stochastic growth and another for stochastic discounting. For future reference, let G be a stochastic growth functional parameterized by (β_g, α_g) . The second will be a stochastic discount functional S parameterized by (β_s, α_s) . The stochastic growth functional captures the evolution of cash flows or other macroeconomic quantities of interest and usually grows exponentially over time. The stochastic discount functional represents marginal valuation and typically decays exponentially.

2.2 Perturbations

To compute elasticities we evaluate expectations of perturbations to multiplicative functionals. The perturbations alter the paths of the functionals while retaining the multiplicative Markov structure and will be used in Section 3 to compute exposure and price elasticities.

A perturbation to M is $MH(\mathbf{r})$, where we parameterize $H(\mathbf{r})$ using a pair $(\beta_h(x, \mathbf{r}), \mathbf{r}\alpha_d(x))$ with $\beta_h(x, 0) = 0$. The function $\alpha_d(x)$ defines the direction of risk exposure. Thus

$$\log H_t(\mathbf{r}) = \int_0^t \beta_h(X_u, \mathbf{r}) du + \mathbf{r} \int_0^t \alpha_d(X_u) \cdot dW_u.$$

In Section 3 we discuss economic motivation that guides the choice of the drift term β_h . As \mathbf{r} declines to zero, the perturbed process $MH(\mathbf{r})$ converges to M . Let

$$\beta_d(x) = \left. \frac{d}{d\mathbf{r}} \beta_h(x, \mathbf{r}) \right|_{\mathbf{r}=0}.$$

Construct the additive functional:

$$D_t = \int_0^t \beta_d(X_u) du + \int_0^t \alpha_d(X_u) \cdot dW_u,$$

which we use to represent the derivative of interest via:

$$\left. \frac{d}{d\mathbf{r}} \log E [M_t H_t(\mathbf{r}) | X_0 = x] \right|_{\mathbf{r}=0} = \frac{E [M_t D_t | X_0 = x]}{E [M_t | X_0 = x]}. \quad (1)$$

Hansen and Scheinkman (2009b) provide a formal derivation including certain regularity conditions that justify this formula. Formula (1) gives an *additive decomposition* through its use of the additive functional D . It what follows we will exploit this additive structure to characterize the contributions of shock exposures at intermediate dates between zero and t .

Recall that an elasticity is the derivative of the logarithm of the outcome with respect to the logarithm of the argument. Our use of the logarithm outside of the expectation is part of the reason we refer to the resulting object as an “elasticity”. We achieve appropriate scaling that supports this interpretation by suitably restricting the magnitude of the direction $\alpha_d(x)$ to satisfy:

$$E [|\alpha_d(X_t)|^2] = 1.$$

2.3 Initial construction of shock elasticities

The perturbation functional H applies to all points in time in the investment horizon between date zero and t . We are also interested in contributions that are localized in time. To accomplish this we seek an integral representation for the derivative:

$$\left. \frac{d}{dr} \log E [M_t H_t(r) | X_0 = x] \right|_{r=0}$$

While the additive functional D has an integral representation including a stochastic integral, we now show how to replace this stochastic integral with a standard integral by computing conditional covariances between M and the stochastic integral component of D .

Given the Brownian information structure we represent M as a stochastic integral:

$$M_t = \int_0^t \chi_{u,t} \cdot dW_u + E(M_t | X_0 = x) \tag{2}$$

which shows how the multiplicative functional is updated in response to shocks.² The coefficients χ give one generalization of an impulse response function familiar from linear time series. For instance, $\chi_{0,t}$ when viewed as a function of t gives the (random) expected response of future values of M to a shock in the next instant conditioned on current information. Of particular interest to us is that

$$E \left[M_t \int_0^t \alpha_d(X_u) \cdot dW_u | X_0 = x \right] = E \left[\int_0^t \alpha_d(X_u) \cdot \chi_{u,t} du | X_0 = x \right]. \tag{3}$$

Asset valuation is often represented in terms of covariances and in this case the essential

²See Theorem 3.4 in Chapter 5 of Revuz and Yor (1991).

covariance is between M and $\int_0^t \alpha_d(X_u)$. Our aim is to produce a more convenient representation for this term.

By construction $(X, \log M)$ is a Markov process. We use the Markov structure of $\log M$ to obtain a formula for the coefficients χ . For a small interval of length h , write

$$E[M_t | \mathcal{F}_{u+h}] - E[M_t | \mathcal{F}_u] = M_{u+h} E\left[\frac{M_t}{M_{u+h}} | X_{u+h}\right] - M_u E\left[\frac{M_t}{M_u} | X_u\right] \quad (4)$$

where we are exploiting the multiplicative construction of M as a function of the Markov process X . When $E\left[\frac{M_t}{M_u} | X_u = x\right]$ is twice continuously differentiable with respect to x we may appeal to Ito's formula in conjunction with the Markov structure to show that the local counterpart to (4) is

$$\chi_{u,t} \cdot dW_u$$

where

$$\begin{aligned} \chi_{u,t} &= E[M_t | M_u, X_u] [\psi(X_u, t - u) + \alpha(X_u)] \\ \psi(x, v) &= \sigma(x)' \left(\frac{\partial}{\partial x} \log E[M_v | X_0 = x] \right). \end{aligned} \quad (5)$$

Section 4 provides an alternative justification based on Malliavin calculus used to implement what is known as the Haussman-Clark-Ocone formula. Substituting formula (5) into (2) and applying the Law of Iterated Expectations gives us the following integral representation of a risk elasticity:

Result 2.1.

$$\frac{1}{t} \frac{d}{dr} \log E[M_t H_t(r) | X_0 = x] \Big|_{r=0} = \frac{1}{t} \frac{E\left[M_t \int_0^t \varepsilon(X_u, t - u) du | X_0 = x\right]}{E[M_t | X_0 = x]}.$$

where

$$\varepsilon(x, v) \doteq \alpha_d(x) \cdot [\psi(x, v) + \alpha(x)] + \beta_d(x) \quad (6)$$

and $\psi(x, v)$ is defined in (5).

We refer to ε as a *shock elasticity function*. From Result 2.1 a risk elasticity over a given investment horizon is an integral over time and a weighted average over states of a shock elasticity function, and thus the shock elasticities are the fundamental building block for risk elasticities. We scale the time integral by the investment horizon t in order to achieve comparability when we explore what happens when we alter t .

In formula (6), α_d parameterizes the local exposure to risk that is being explored and β_d

is determined as a consequence of the nature of the perturbation. In Section 3 we show how economic considerations can guide us in choosing β_d . The coefficient α is the local exposure to risk of the baseline multiplicative functional. Recall that to interpret the logarithmic derivative as an elasticity, we restrict $|\alpha_d(X_t)|^2$ to have a unit expectation so that $\alpha_d(X_t) \cdot dW_t$ has a unit standard deviation scaled by dt . The dependence of ε on the horizon to which the perturbation pertains, that is the dependence on t , is only manifested in the function ψ . The shock elasticity function includes a direct effect captured by α which is the local exposure of $\log M$ to the Brownian increment, and an indirect effect captured by ψ which is constructed from the impulse response function for M .

2.4 Martingale decomposition

We obtain an alternative and convenient representation of (1) by applying a change of measure. This change of measure gives us a characterization of elasticities as the investment horizon becomes large by identifying the long-term shock exposure of M through its martingale component. We construct the change of measure by factoring the multiplicative functional, and we show how to apply this change to our calculations. Our use of a multiplicative factorization differentiates this from commonly used methods of identifying permanent shocks.

Hansen and Scheinkman (2009a) provide sufficient conditions for the existence of a factorization of a multiplicative process M :

$$M_t = \exp(\eta t) \hat{M}_t \frac{e(X_0)}{e(X_t)} \quad (7)$$

where \hat{M} is a multiplicative martingale and e is a strictly positive, smooth function of the Markov state. This function represents the most durable dominant component of the transient dynamics of M . The parameter η is a long-term growth or decay rate. We use the martingale \hat{M} to define a new probability measure on the original probability space. The multiplicative property of \hat{M} ensures that X remains Markov in the new probability space. While this factorization may not be unique, there is only one such factorization in which the change in measure imposes stochastic stability.³

Our factorization is distinct from that of Ito and Watanabe (1965). The Ito and Watanabe (1965) factorization for a multiplicative supermartingale results in the product of a local martingale and a decreasing functional. This factorization delivers the Markov counterpart to the risk neutral transformation used extensively in mathematical finance when it is applied

³The notion of stochastic stability that interests us is that conditional expectations of functions of the Markov state converge to their unconditional counterparts as the forecast horizon is increased.

to a stochastic discount factor functional. In this case the decreasing functional M^d is

$$M_t^d = \exp \left[- \int_0^t \varrho(X_u) du \right]$$

where ϱ is the instantaneous interest rate. State dependence in the decreasing component makes it less valuable as a device to characterize risk price dynamics because even locally deterministic variation in instantaneous interest rates induces risk adjustments for cash flows over finite time intervals. This leads us instead to extract a long-term growth or discount rate η as in (7).

Parameterizing M by (β, α) , Girsanov's Theorem ensures the increment dW_t can be written as:

$$dW_t = [\alpha(X_t) + \nu(X_t)] dt + d\hat{W}_t. \quad (8)$$

Here $\nu(x)$ is the exposure of $\log e(x)$ to dW_t :

$$\nu = \sigma' \left[\frac{\partial \log e}{\partial x} \right]$$

and \hat{W} is a Brownian motion under the alternative probability measure $\hat{\cdot}$. Alternatively, $\alpha + \nu$ is the shock exposure of the logarithm of martingale \hat{M} .

To use this factorization in practice, we must compute e and η . Hansen and Scheinkman (2009a) show how to accomplish this. Solve

$$E [M_t e(X_t) | X_0 = x] = \exp(\eta t) e(x)$$

for any t where e is strictly positive. This is a (principal) eigenfunction problem, and since it holds for any t , it can be localized by computing

$$\lim_{t \rightarrow 0} \frac{E [M_t e(X_t) | X_0 = x] - \exp(\eta t) e(x)}{t} = 0$$

which gives an equation in e and η to be solved. The local counterpart to this equation is

$$\mathbb{B}e = \eta e \quad (9)$$

where

$$\mathbb{B}e(x) = \frac{d}{dt} E [M_t e(X_t) | X_0 = x] \Big|_{t=0}$$

It can be shown that for a diffusion model, if f is smooth,

$$\mathbb{B}f = \left(\beta + \frac{1}{2}|\alpha|^2 \right) f + (\sigma\alpha + \mu) \cdot \frac{\partial f}{\partial x} + \frac{1}{2} \text{trace} \left(\sigma\sigma' \frac{\partial^2 f}{\partial x \partial x'} \right).$$

2.5 Elasticities under the change of measure

We use the alternative probability measure to absorb the martingale component of the multiplicative functional in our formula (7). Under the change of measure, the drift component of the additive functional D picks up the diffusion term of this martingale component

$$\hat{\beta}_d = \beta_d + \alpha_d \cdot (\alpha + \nu). \quad (10)$$

In Section 3, we provide economic motivation for the choice of the perturbation H and thus for the coefficients (β_d, α_d) that restricts specific functionals to be martingales under the original measure. Equation (10) shows how the change of measure is compensated in the drift term of the perturbation.

Writing \hat{E} for the expectation operator under the change in measure induced by \hat{M} , we obtain:

$$\frac{d}{dr} \log E [M_t H_t(r) | X_0 = x] \Big|_{r=0} = \frac{\hat{E} \left[\hat{e}(X_t) \int_0^t \varepsilon(X_u, t-u) du | X_0 = x \right]}{\hat{E} [\hat{e}(X_t) | X_0 = x]}$$

where $\hat{e} = \frac{1}{e}$. Using the alternative probability measure we find that

$$\begin{aligned} \psi(x, v) &= \sigma(x)' \left(\frac{\partial}{\partial x} \log E [M_v | X_0 = x] \right) \\ &= \phi(x, v) - \phi(x, 0) \end{aligned}$$

where

$$\phi(x, v) = \sigma(x)' \left(\frac{\partial}{\partial x} \log \hat{E} [\hat{e}(X_v) | X_0 = x] \right). \quad (11)$$

This leads us to reformulate Result 2.1 under the alternative probability measure:

Result 2.2.

$$\frac{1}{t} \frac{d}{dr} \log E [M_t H_t(r) | X_0 = x] \Big|_{r=0} = \frac{1}{t} \frac{\hat{E} \left[\hat{e}(X_t) \int_0^t \varepsilon(X_u, t-u) du | X_0 = x \right]}{\hat{E} [\hat{e}(X_t) | X_0 = x]}.$$

where

$$\varepsilon(x, v) \doteq \alpha_d(x) \cdot [\phi(x, v) + \nu(x) + \alpha(x)] + \beta_d(x) \quad (12)$$

and $\phi(x, v)$ is defined in (11).

In this formula we use the fact that $\nu(x) = -\phi(x, 0)$ where ν captures how the dominant eigenfunction e is exposed to shocks. The shock elasticity $\varepsilon(x, t)$ is unaffected by the change of measure but the contribution $\alpha_d \cdot (\nu + \alpha)$ coming from the martingale component \hat{M} is singled out to the drift term of the additive functional D , as shown in formula (10).

The limiting shock elasticities are given by

$$\begin{aligned} \lim_{v \downarrow 0} \varepsilon(x, v) &= \alpha_d(x) \cdot \alpha(x) + \beta_d(x) \\ \lim_{v \uparrow \infty} \varepsilon(x, v) &= \alpha_d(x) \cdot [\nu(x) + \alpha(x)] + \beta_d(x) \end{aligned}$$

where the latter formula follows from the fact that X is stochastically stable under the change of measure. In this formula α_d defines the direction for the exposure to be valued, α is the local exposure of $\log M$ to the shock increment dW (and to the increment $d\hat{W}$), and as we remarked earlier, $\nu + \alpha$ is the exposure of $\log M$ to the shock increments. We will have more to say about the role of β_d later in our analysis. The dependence on the investment horizon is captured by $\phi(x, v)$.

To interpret the contribution ϕ to ε at intermediate dates, note that

$$\hat{e}(X_t) = \int_0^t \hat{E} [\hat{e}(X_{t-u}) | X_0 = x] \phi(x, t-u) d\hat{W}_u + \hat{e}(X_0),$$

which gives a moving-average representation with state dependent coefficients. In particular the contribution

$$\hat{E} [\hat{e}(X_t) | X_0 = x] \phi(x, t)$$

gives a measure of the response of $\hat{e}(X_t)$ to a shock at date zero.

Result 2.2 also has implications for the valuation of the exposure to shocks that occur in the future. Our shock elasticities exploit the local normality built into the diffusion specification, but as we shift the date of the shock forward in time, there is an additional role for the distribution of the state dynamics. Consider the exposure to a shock at date τ , which has implications for valuation of payoffs maturing from date τ forward. Its impact will be realized through a distorted conditional expectation. For the current state x and the investment horizon $t + \tau$ we construct:

$$\varepsilon(x, t; \tau) = \frac{\hat{E} [\hat{e}(X_{t+\tau}) \varepsilon(X_\tau, t) | X_0 = x]}{\hat{E} [\hat{e}(X_{t+\tau}) | X_0 = x]} \quad (13)$$

Since the process X is stochastically stable under the change of measure, the limiting version

of formula (13) as the shock date τ is shifted to the future is

$$\varepsilon(t; \infty) = \frac{\hat{E} [\hat{\varepsilon}(X_t)\varepsilon(X_0, t)]}{\hat{E} [\hat{\varepsilon}(X_t)]} \quad (14)$$

which is independent of τ and x but continues to depend on t , the time between the shock and the payoff horizon.

3 Price and exposure elasticities

Risk premia come from two sources, exposure to risk and the price of that exposure. This leads us to construct two types of elasticities: exposure and price elasticities. Consider a parameterized family of cash flows $GH(r)$ to be valued. Exposure elasticities measure how changes in an expected growth functional $GH(r)$ are altered as we change the exposure parameterized by r . Price elasticities measure how changes in a corresponding expected return are altered as we change r and include a contribution from the stochastic discount factor functional S . In this section we define both elasticities and specify formally the perturbations used in the constructions. Consistent with our development in Section 2 we distinguish between risk elasticities and their instantaneous counterparts, shock elasticities.

3.1 Risk-price elasticity

Consider the expected return over an investment horizon t subject to a perturbation to the cash flow:

$$\frac{E [G_t H_t(r) | X_0 = x]}{E [S_t G_t H_t(r) | X_0 = x]}.$$

Taking logarithms, scaling by the payoff horizon, and differentiating with respect to r gives

$$\pi(x, t) = \frac{1}{t} \frac{d}{dr} \log E [G_t H_t(r) | X_0 = x] \Big|_{r=0} - \frac{1}{t} \frac{d}{dr} \log E [S_t G_t H_t(r) | X_0 = x] \Big|_{r=0}, \quad (15)$$

which is the *risk-price elasticity* associated with direction $\alpha_d(x)$ that is implied by the construction of the perturbation H . By using expected returns to measure a risk price, we follow an approach that is typical in one-period (in discrete time) or instantaneous (in continuous time) valuation problems. The returns are themselves constructed to have a unit price in terms of a consumption numeraire, but their expectations are sensitive to changes in the risk exposure.

The risk-price elasticity consists of two components. We call the first term a *risk-exposure elasticity* because it captures the sensitivity of expected cash flows to risk exposure. The

second term, which we call the *risk-value elasticity*, includes the sensitivity of the cash flow value to changes in the risk exposure.⁴ In contrast to familiar risk premia, the risk price elasticities express the rewards to *marginal* changes in risk exposure in a particular direction. In the special case of lognormal models, marginal and average rewards to risk exposure coincide, and the risk premium can be expressed as the risk-price elasticity in the direction G multiplied by the appropriate quantity of risk exposure, but nonlinearity in the Markov evolution typically overturns this result as we illustrate in the examples in Section 5.

3.2 Martingale perturbations

One convenient choice of the perturbation $H(r)$ for building elasticities is to restrict it to be a (local) martingale. In this way we deliberately abstract from augmenting the cash-flow dynamics by the choice of the perturbation. To impose the martingale restriction we set

$$\beta_h(x, r) = -\frac{1}{2}r^2|\alpha_d(x)|^2.$$

In this case $\beta_d = 0$ and

$$\hat{\beta}_d = \alpha_d \cdot (\alpha + \nu).$$

The input in formula (1) then becomes an additive (local) martingale under the original probability measure:

$$D_t = \int_0^t \alpha_d(X_u) \cdot dW_u.$$

With martingale perturbations, we essentially recover impulse responses as shock elasticities. One construction of an impulse response function is $\chi_{0,t}$ used to represent M_t as

$$M_t = \int_0^t \chi_{u,t} \cdot dW_u + E[M_0|X_0 = x]$$

where $\chi_{0,t}$ measures how M_t responds to a shock modeled as a Brownian increment at date zero conditional on date zero information. Then

$$\frac{1}{t} \frac{d}{dr} \log E[M_t H_t(r)|X_0 = x] \Big|_{r=0} = \frac{1}{t} \frac{E \left[\int_0^t \alpha_d(X_u) \cdot \chi_{u,t} du | X_0 = x \right]}{E[M_t|X_0 = x]}. \quad (16)$$

The term $\alpha_d(x) \cdot \chi_{0,t}$ measures the expected response of M_t to a shock $\alpha_d(x) \cdot dW_0$. Formula (5)

⁴This approach to pricing risk of cash flows with stochastic growth components follows Hansen et al. (2008), Hansen and Scheinkman (2009a), and Hansen (2009). The priced cash flows are sometimes referred to as zero coupon equity (see Wachter (2005) or Lettau and Wachter (2007)), that is a claim to a single random payoff at a point in time t .

in Section 2 represents $\chi_{0,t}$ as:

$$\begin{aligned}\chi_{0,t} &= E[M_t|X_0 = x] [\psi(x, t) + \alpha(x)] \\ \psi(x, t) &= \sigma(x)' \left(\frac{\partial}{\partial x} \log E[M_t|X_0 = x] \right)\end{aligned}$$

where the scale factor $E[M_t|X_0 = x]$ is also present in the denominator of the right-hand side of (16). Later we will draw connections to other ways of constructing impulse response functions.

Recall from formula (15) that the risk-price elasticity has two components, which we now consider in turn. The first term uses the multiplicative functional $M = G$ and results in a risk-exposure elasticity. The second one uses $M = V = SG$ and results in a risk-value elasticity. The value component interacts the exposure of the cash flow to risk and the price of that risk as reflected by the marginal investor. By forming the difference we obtain a risk-price elasticity.

We use Result 2.2 to represent the risk-exposure elasticity as:

$$\frac{1}{t} \frac{d}{dr} \log E[G_t H_t(r)|X_0 = x] \Big|_{r=0} = \frac{1}{t} \frac{\hat{E}_g \left[\hat{e}_g(X_t) \int_0^t \varepsilon_g(X_u, t-u) du | X_0 = x \right]}{\hat{E}_g [\hat{e}_g(X_t) | X_0 = x]}.$$

We obtain the distorted expectation and the function \hat{e} subscripted by g from the multiplicative factorization of G and

$$\varepsilon_g(x, t) = \alpha_d(x) \cdot [\psi_g(x, t) + \alpha_g(x)].$$

We repeat this calculation for $M = V$ and construct the risk-value elasticity

$$\frac{1}{t} \frac{\hat{E}_v \left[\hat{e}_v(X_t) \int_0^t \varepsilon_v(X_u, t-u) du | X_0 = x \right]}{\hat{E}_v [\hat{e}_v(X_t) | X_0 = x]}$$

with the corresponding the shock-value elasticity function

$$\varepsilon_v(x, t) = \alpha_d(x) \cdot [\psi_v(x, t) + \alpha_g(x) + \alpha_s(x)].$$

Thus using Result 2.2, we rewrite the risk-price elasticity (15) as

$$\pi(x, t) = \frac{1}{t} \frac{\hat{E}_g \left[\hat{e}_g(X_t) \int_0^t \varepsilon_g(X_u, t-u) du | X_0 = x \right]}{\hat{E}_g [\hat{e}_g(X_t) | X_0 = x]} - \frac{1}{t} \frac{\hat{E}_v \left[\hat{e}_v(X_t) \int_0^t \varepsilon_v(X_u, t-u) du | X_0 = x \right]}{\hat{E}_v [\hat{e}_v(X_t) | X_0 = x]} \quad (17)$$

where the subindices g and v index terms obtained in the martingale factorizations of the functionals G and V , respectively. Equation (17) is an integral representation of the risk-price elasticity.

Collecting the two shock elasticities, we define the *shock-price elasticity function* as

$$\varepsilon_p(x, t) = \varepsilon_g(x, t) - \varepsilon_v(x, t) = \alpha_d(x) \cdot [\psi_g(x, t) - \psi_v(x, t) - \alpha_s(x)]. \quad (18)$$

While this construction is of interest for studying the impact of a shock over the next instant, the two components must be treated separately when studying the impact of shocks in the future dates. While the exposure and value elasticities over an investment interval t are distorted expectations of integrals of the corresponding shock elasticities, this is not the case for the price elasticity once we change measures. The multiplicative functionals $M = G$ and $M = SG$ will typically have different martingale components so two different changes of measure come into play in the construction of risk-price elasticities. Following Hansen (2009), we consider next an alternative approach that avoids this complication.

3.3 Martingale growth functionals

The alternative approach suggested by Hansen (2009) avoids the construction of two separate components. Instead we build G to be a multiplicative martingale. To enforce this restriction we set

$$\beta_g(x) = -\frac{1}{2} |\alpha_g(x)|^2.$$

The functional G could be the martingale component of a baseline macroeconomic growth functional or of some other multiplicative cash flow. By construction, the expected cash flow is identically one and the source of the risk price dynamics is the stochastic discount factor functional. Further suppose that $GH(\mathbf{r})$ is also a martingale implying that:

$$E [G_t H_t(\mathbf{r}) | X_0 = x] = 1$$

for all r . This martingale restriction is satisfied when

$$\beta_h(x, r) - \frac{1}{2}|\alpha_g(x)|^2 = -\frac{1}{2}|\alpha_g(x) + r\alpha_d(x)|^2.$$

Differentiating with respect to r yields

$$\beta_d(x) = \left. \frac{d}{dr} \beta_h(x, r) \right|_{r=0} = -\alpha_d(x) \cdot \alpha_g(x).$$

As a consequence, the additive functional D now contains a drift term

$$D_t = -\int_0^t \alpha_d(X_u) \cdot \alpha_g(X_u) du + \int_0^t \alpha_d(X_u) \cdot dW_u.$$

This results in the following measure for the risk-price elasticity

$$\pi(x, t) = -\frac{1}{t} \frac{E[S_t G_t D_t | X_0 = x]}{E[S_t G_t | X_0 = x]}$$

because of the martingale construction of the cash-flow dynamics. The exposure elasticities for the cash flow are zero by construction. The shock-price elasticity function is now given by:

$$\varepsilon_p(x, t) = -\varepsilon_v(x, t) = -\alpha_d(x) \cdot [\psi_v(x, t) + \alpha_s(x)]$$

and the risk-price elasticity for investment horizon t is

$$\pi(x, t) = \frac{1}{t} \frac{\hat{E}_v \left[\hat{e}_v(X_t) \int_0^t \varepsilon_p(X_u, t-u) du | X_0 = x \right]}{\hat{E}_v [\hat{e}_v(X_t) | X_0 = x]}.$$

3.4 Limiting elasticities

To relate our analysis to previous pricing characterizations consider the local and long-horizon limits of the shock price elasticity function. Since $\psi(x, 0) = 0$ by construction, the local price elasticity is

$$\pi(x, 0) \equiv \varepsilon_p(x, 0) = -\alpha_d(x) \cdot \alpha_s(x)$$

which implies that $-\alpha_s$ is the local price vector for exposure α_d . This reproduces the standard continuous-time pricing of Brownian increments by the exposure of the stochastic discount factor to shocks. The change of measure allows us to conveniently represent the long-horizon elasticities as featured by Hansen (2009). Since the Markov process is stochastically stable

under the change of measure, $\phi(x, t)$ in equation (11) vanishes as $t \rightarrow \infty$, and the large t limit for the price elasticity is:

$$\varepsilon_p(x, \infty) = \alpha_d(x) \cdot [\nu_g(x) - \nu_v(x) - \alpha_s(x)].$$

This limit includes contributions from the exposure of the dominant eigenfunctions ν for growth and valuation to the Brownian increment. Due to the permanent nature of the shocks to growth rates and discount rates, this long-horizon elasticity does not vanish and in general still depends on x .

4 Hausmann-Clark-Ocone formula

In our initial development we built a moving-average representation for the multiplicative functional with state-dependent coefficients. This formula can be viewed as a special case of the Hausmann-Clark-Ocone formula because the latter formula can be justified under weaker smoothness conditions. For example see Hausmann (1979). In this section provide an explicit discussion of the Hausman-Clark-Ocone formula and its relation to Malliavin calculus. This digression is not essential to follow the remainder of our paper. We include it for readers familiar with the continuous-time tools used in mathematical finance including the Malliavin derivative.

Following the seminal paper Ocone and Karatzas (1991), results from Malliavin calculus have been used to derive expressions for asset prices, their volatilities, optimal allocations or portfolios, in particular in models with more sophisticated intertemporal dependencies.⁵ Consider the following perturbations to the Brownian motion between date zero and date t . Let q be a function in $L_2^k[0, t]$, that is

$$\int_0^t |q(v)|^2 dv < \infty.$$

The perturbed process is:

$$W_u + rQ_u, \quad 0 \leq u \leq t$$

where $Q_u = \int_0^u q(v)dv$, and $r \in \mathbb{R}$. Recall that we can identify each path of a Brownian motion in $[0, t]$ with an element of $\Omega = C_0([0, t], \mathbb{R}^k)$, the set of continuous \mathbb{R}^k -valued functions starting at 0. Given a random variable Φ defined on Ω with a finite second moment, we are interested in the derivative of $\Phi(W + rQ)$ with respect to r . The Malliavin derivative is a

⁵See Detemple and Zapatero (1991) for another early example of this literature.

process $D_u\Phi(W)$ in $L_2(\Omega \times [0, t])$ that is motivated by the following representation:⁶

$$\lim_{r \rightarrow 0} \frac{\Phi(W + rQ) - \Phi(W)}{r} = \int_0^t D_u\Phi(W) \cdot q(u) du. \quad (19)$$

The value of the Malliavin derivative at u quantifies the contribution of dW_u to Φ . This contribution will, in general, depend on the entire Brownian path from 0 to t .

Fix an initial condition x and a time t and consider the random variable Φ defined by

$$\Phi(W) = M_t$$

where $(X, \log M)$ solves

$$\begin{aligned} dX_u &= \mu(X_u)du + \sigma(X_u)dW_u \\ d\log M_u &= \beta(X_u)du + \alpha(X_u) \cdot dW_u. \end{aligned}$$

Here, X is an n -dimensional process, W is a k -dimensional Brownian motion, and M a multiplicative functional. Given that the multiplicative functional is built from the Markov process, it is convenient to construct the Malliavin derivative in three steps. In the first step we compute the $\mathbb{R}^{n \times k}$ -valued process $D_u X_\tau = Y_\tau$. If the functions μ and σ are smooth and with bounded derivatives then the random variable X_τ is in the domain of the Malliavin derivative. This derivative is defined by the solution to:

$$dY_\tau = \partial\mu(X_\tau)Y_\tau d\tau + \sum_i \partial\sigma^i(X_\tau)Y_\tau dW_\tau^{[i]}$$

for $\tau \geq 0$ with the initial condition $Y_0 = I$. Here, ∂F denotes the $n \times n$ Jacobian matrix of an \mathbb{R}^n -valued function F , σ^i is the i -th column of the matrix σ and $W_\tau^{[i]}$ is the i -th entry of W_τ . Then

$$D_u X_\tau = Y_\tau(Y_u)^{-1}\sigma(X_u).$$

for $\tau \geq u$.⁷

In the second step we compute $D_u \log M_t$. If the functions β and α are smooth, then the

⁶The construction of the Malliavin derivative usually starts by considering a subset of random variables called the Wiener polynomials and defining the Malliavin derivative using equation (19). The Malliavin derivative is then extended to a larger class of random variables with finite second moments using limits. Equation (19) does not necessarily hold for every random variable which has a Malliavin derivative.

⁷The term $(Y_u)^{-1}$ in effect reinitializes the process Y to be the identity at $\tau = u$ and the multiplication by $\sigma(X_u)$ accounts for the impact of dW_u at $\tau = u$.

random variable $\log M_t$ is in the domain of the Malliavin derivative. This derivative

$$D_u \log M_t = \int_u^t \partial\beta(X_\tau) D_u X_\tau d\tau + \sum_i \int_u^t \partial\alpha^{[i]}(X_\tau) D_u X_\tau dW_\tau^{[i]} + \alpha(X_u)'$$

has the same dimension as the vector α' and $\alpha^{[i]}$ is the i -th element of α . This formula is justified as an application of the chain rule provided that $\log M_t$ has a finite second moment and the right-hand side is in $L_2(\Omega \times [0, t])$.⁸

Finally, in the third step we compute

$$D_u \Phi(W) = D_u M_t = M_t D_u \log M_t$$

by again applying the chain rule where M_t has a finite second moment and the process $\{M_t D_u \log M_t : 0 \leq u \leq t\}$ is in $L_2(\Omega \times [0, t])$.

The Haussmann-Clark-Ocone formula provides a representation of the integrator χ in equation (2) in terms of a Malliavin derivative:⁹

$$(\chi_{u,t})' = E [D_u \Phi(W) | \mathcal{F}_u],$$

and thus¹⁰

$$M_t = \int_0^t E [D_u \Phi(W) | \mathcal{F}_u] dW_u + E (M_t | X_0 = x).$$

Furthermore,

$$\frac{(\chi_{u,t})'}{E [M_t | \mathcal{F}_u]} = \frac{E \left[\frac{D_u \Phi(W)}{M_u} | \mathcal{F}_u \right]}{E \left[\frac{M_t}{M_u} | \mathcal{F}_u \right]} = \frac{E \left[\frac{D_u M_t}{M_u} | X_u \right]}{E \left[\frac{M_t}{M_u} | X_u \right]}$$

where the last equality follows because $\frac{D_u M_t}{M_u}$ and $\frac{M_t}{M_u}$ depend only on the Markov process X between dates u and t . This leads us to represent the function ψ via¹¹

$$\psi(x, t - u)' = \frac{E \left[\frac{D_u M_t}{M_u} | X_u = x \right]}{E \left[\frac{M_t}{M_u} | X_u = x \right]} - \alpha(x)'$$

⁸See Len et al. (2003) Lemma 2.1.

⁹For a statement of this formula and the results concerning the Malliavin derivative of functions of a Markov diffusion see, for instance, Fournié et al. (1999), pages 395 and 396.

¹⁰Haussmann (1979) gives formulas for Markov dynamics for more general functions Φ .

¹¹Gourieroux and Jasiak (2005) suggest basing impulse response functions on the pathwise contribution to changing a shock at a given date. This leads them to explore more general distributional consequences of a shock. The Malliavin derivative is the continuous-time counterpart and depends on the entire shock process up to date t .

In order to replicate formula (5) from Section 2, we exchange orders of differentiation and expectation:¹²

$$\begin{aligned} E[D_u M_t | \mathcal{F}_u] &= D_u E[M_t | \mathcal{F}_u] = D_u M_u E\left[\frac{M_t}{M_u} | \mathcal{F}_u\right] = \\ &= E\left[\frac{M_t}{M_u} | \mathcal{F}_u\right] M_u \alpha(X_u)' + M_u \left(\frac{\partial}{\partial x} E\left[\frac{M_t}{M_u} | X_u\right]\right)' \sigma(X_u) = \\ &= E[M_t | \mathcal{F}_u] \left[\alpha(X_u)' + \left(\frac{\partial}{\partial x} \log E\left[\frac{M_t}{M_u} | X_u\right]\right)' \sigma(X_u)\right]. \end{aligned}$$

Thus

$$\frac{E\left[\frac{D_u M_t}{M_u} | X_u = x\right]'}{E\left[\frac{M_t}{M_u} | X_u = x\right]} - \alpha(x) = \sigma(x)' \left(\frac{\partial}{\partial x} \log E\left[\frac{M_t}{M_u} | X_u = x\right]\right)$$

which agrees with the right-hand side of formula (5).

In defining the Malliavin derivative, we introduced deterministic drift distortions of Q . Bismut (1981) uses bounded drift distortions that can be measurable functions of the Brownian path and constructs an alternative proof of a representation like that in (3).¹³ Our approach and that of Hansen and Scheinkman (2009b) is very closely related to that of Bismut (1981). Consider first the case in which $H(\mathbf{r})$ is a parameterized martingale perturbation as in Section 3.2. While we use this parameterized perturbation to change the risk exposure, it is also associated with a change in probability measure for which W has drift distortion $\mathbf{r} \int_0^t \alpha_d(X_u) du$. This is the proof strategy adopted in Bismut (1981).¹⁴ For the case considered in Section 3.3 in which perturbations are restricted so that $GH(\mathbf{r})$ is a parameterized family of martingales, Hansen and Scheinkman (2009b) use G to change probability measures and then treat $H(\mathbf{r})$ as a martingale under this change of measure. Thus there is also a close connection to the approach of Bismut (1981) for our second choice of perturbations. The restrictions imposed in Bismut (1981) are too stringent for our purposes, but Hansen and Scheinkman (2009b) give weaker conditions for these results.

5 Recursive utility specifications of investor preferences

In this and the next section, we compute elasticities for model economies taken from the existing asset pricing literature. Before studying asset pricing implications, we show how to compute the shock elasticities under an *affine model* that nests a model with lognormal

¹²For instance, see Øksendal (1997) Proposition 5.6 in Chapter 5.

¹³See formula (2.43) in Bismut (1981).

¹⁴See equation (2.4) in the proof of Theorem 2.1 in Bismut (1981).

dynamics commonly used in VAR analysis, but allows for state-dependent volatilities. In this section we use the affine specification as a reduced form for example economies in which investors have preferences represented by a power utility function or preferences represented by a recursive utility function of the type suggested by Kreps and Porteus (1978) and Epstein and Zin (1989). As in the long-run risk literature (see Bansal and Yaron (2004)), we postulate consumption dynamics that contain a small predictable component in macroeconomic growth and stochastic volatility. We study how the consumption dynamics in conjunction with investor's preferences influence the risk-price and shock-price dynamics, extending previous work of Hansen et al. (2008) and Hansen (2009).¹⁵

5.1 Affine dynamics with stochastic volatility

Suppose that the state vector is $X = (X^{[1]}, X^{[2]})'$ where $X^{[1]}$ is an n -dimensional vector and $X^{[2]}$ a scalar. Its dynamics are specified by

$$\mu(x) = \begin{bmatrix} \bar{\mu}_{11} & \bar{\mu}_{12} \\ 0 & \bar{\mu}_{22} \end{bmatrix} \begin{bmatrix} x^{[1]} - \iota_1 \\ x^{[2]} - \iota_2 \end{bmatrix} \quad \sigma(x) = \sqrt{x^{[2]}}\bar{\sigma} = \sqrt{x^{[2]}} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{bmatrix}, \quad (20)$$

where $\bar{\mu}_{11}$ and $\bar{\mu}_{12}$ are $n \times n$ and $n \times 1$ matrices, $\bar{\mu}_{22}$ is a scalar, and $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are $n \times k$ and $1 \times k$ matrices, respectively. Consider a multiplicative functional parameterized by

$$\beta(x) = \bar{\beta}_0 + \bar{\beta}_1 \cdot (x^{[1]} - \iota_1) + \bar{\beta}_2(x^{[2]} - \iota_2) \quad \alpha(x) = \sqrt{x^{[2]}}\bar{\alpha}. \quad (21)$$

This specification of the dynamics allows for a predictable component in the multiplicative functional, modeled by $X^{[1]}$, and for stochastic volatility, modeled by the scalar process $X^{[2]}$. Our variance process $X^{[2]}$ stays strictly positive; we prevent it from being pulled to zero by imposing the restrictions $\iota_2 > 0$ and $\bar{\mu}_{22} + \frac{1}{2}|\bar{\sigma}_2|^2 < 0$. To guarantee the existence of a stationary distribution, we assume that $\bar{\mu}_{11}$ has eigenvalues with strictly negative real parts. The parameters ι_1 and ι_2 are the unconditional means for $X^{[1]}$ and $X^{[2]}$ in the stationary distribution. Setting $\bar{\sigma}_2 = 0$ and $X^{[2]} \equiv 1$ reduces the dynamics to a lognormal model familiar from the VAR literature.

This model specification implies two useful properties in calculating shock elasticities. First, conditional expectations are loglinear in the state variables, with time-dependent coefficients given as solutions to a set of first-order ordinary differential equations. Second,

¹⁵Grasselli and Tebaldi (2004) analyze the class of affine term structure models from a related perspective. They derive explicit formulas for the impulse response function of the factor process under the affine dynamics by utilizing a link between the known solutions for bond prices and the Malliavin derivative of the factor process.

the principal eigenfunction associated with the martingale decomposition is loglinear in the state variables. In fact, $e(x) = \exp(\lambda_1 \cdot x^{[1]} + \lambda_2 x^{[2]})$.

To find the eigenvalue and eigenfunction for multiplicative functional M of form (21), we note that equation (9) specialized to this stochastic specification implies a pair of conditions that determine λ :

$$\begin{aligned} 0 &= \bar{\beta}_1 + (\bar{\mu}_{11})' \lambda_1 \\ 0 &= \bar{\beta}_2 + (\bar{\mu}_{12})' \lambda_1 + \bar{\mu}_{22} \lambda_2 + \frac{1}{2} |\bar{\alpha}' + (\lambda_1)' \bar{\sigma}_1 + \lambda_2 \bar{\sigma}_2|^2. \end{aligned} \quad (22)$$

Additionally, the associated eigenvalue is given by

$$\eta = \bar{\beta}_0 - (\iota_1)' [\bar{\beta}_1 + (\bar{\mu}_{11})' \lambda_1] - \iota_2 [\bar{\beta}_2 + (\bar{\mu}_{12})' \lambda_1 + \bar{\mu}_{22} \lambda_2].$$

Since equation (22) has in general multiple solutions, we follow Hansen and Scheinkman (2009a) and choose the solution that is associated with the smallest eigenvalue. This solution is the one that leads to stable dynamics of the Markov process X .

The martingale \hat{M} is also a multiplicative functional with

$$\begin{aligned} \hat{\alpha}(x) &= \sqrt{x^{[2]}} [\bar{\alpha} + (\bar{\sigma}_1)' \lambda_1 + (\bar{\sigma}_2)' \lambda_2] \\ \hat{\beta}(x) &= -\frac{1}{2} |\hat{\alpha}(x)|^2. \end{aligned}$$

Under the change of measure

$$dW_t = \hat{\alpha}(X_t) dt + d\hat{W}_t,$$

where \hat{W} is a multivariate standard Brownian motion under the probability measure induced by \hat{M} . With this change of measure, X remains a Markov process with drift coefficient

$$\mu(x) + \sqrt{x^{[2]}} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{bmatrix} \hat{\alpha}(x).$$

The functional form for the dynamic evolution is the same as the original specification but the parameter values differ.

By exploiting the calculations from Duffie and Kan (1994), Hansen (2009) shows that for a multiplicative functional M parameterized by (21),

$$E[M_t | X_0 = x] = \exp[\theta_0(t) + \theta_1(t) \cdot x^{[1]} + \theta_2(t) x^{[2]}]$$

where the $\theta_i(t)$ coefficients satisfy the following set of ordinary differential equations, each

with initial condition $\theta_i(0) = 0$:

$$\begin{aligned}
\frac{d}{dt}\theta_1(t) &= \bar{\beta}_1 + (\bar{\mu}_{11})'\theta_1(t) \\
\frac{d}{dt}\theta_2(t) &= \bar{\beta}_2 + (\bar{\mu}_{12})'\theta_1(t) + \bar{\mu}_{22}\theta_2(t) + \frac{1}{2}|\bar{\alpha}' + \theta_1(t)'\bar{\sigma}_1 + \theta_2(t)\bar{\sigma}_2|^2 \\
\frac{d}{dt}\theta_0(t) &= \bar{\beta}_0 - (\iota_1)'[\bar{\beta}_1 + (\bar{\mu}_{11})'\theta_1(t)] - \iota_2[\bar{\beta}_2 + (\bar{\mu}_{21})'\theta_1(t) + \bar{\mu}_{22}\theta_2(t)].
\end{aligned} \tag{23}$$

Since

$$\frac{\partial}{\partial x} \log E[M_t | X_0 = x] = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix},$$

it follows from Result 2.1 that the shock elasticity is:

$$\varepsilon(x, t) = \beta_d(x) + \alpha_d(x) \cdot [(\bar{\sigma}_1)'\theta_1(t) + (\bar{\sigma}_2)'\theta_2(t) + \bar{\alpha}] \sqrt{x^{[2]}}$$

where α_d selects the direction of the shock, and β_d is a function that is determined according to the particular application characterized in Section 3. Since $X^{[2]}$ has mean ι_2 under the stationary distribution, we normalize the coefficient vector α_d so that

$$|\alpha_d(x)|^2 = \frac{1}{\iota_2}.$$

Notice that the first two components to equation (22) for λ_1 and λ_2 give the stationary levels for $\theta_1(t)$ and $\theta_2(t)$. In fact λ_1 and λ_2 are the limit points of $\theta_1(t)$ and $\theta_2(t)$. Thus to represent large t behavior, we write

$$\varepsilon(x, t) = \beta_d(x) + \alpha_d(x) \cdot [(\bar{\sigma}_1)'[\theta_1(t) - \lambda_1]\sqrt{x^{[2]}} + (\bar{\sigma}_2)'[\theta_2(t) - \lambda_2]\sqrt{x^{[2]}} + \hat{\alpha}(x)]$$

with a large t limit given by

$$\varepsilon(x, \infty) = \beta_d(x) + \alpha_d(x) \cdot \hat{\alpha}(x).$$

Thus the drift distortion $\hat{\alpha}$ in the change of measure is also a central component to the limiting shock elasticity, consistent with our general analysis. The transient contribution to the elasticity satisfies

$$\frac{\partial}{\partial x} \log \hat{E} \left[\exp \left(-\lambda_1 \cdot X_t^{[1]} - \lambda_2 X_t^{[2]} \right) | X_0 = x \right] = \begin{bmatrix} \theta_1(t) - \lambda_1 \\ \theta_2(t) - \lambda_2 \end{bmatrix},$$

where the expectation is computed under the change of measure.

The differential equation for $\theta_1(t)$ in (23) yields the solution

$$\theta_1(t) = (\exp [(\bar{\mu}_{11})'t] - I) [(\bar{\mu}_{11})']^{-1} \bar{\beta}_1,$$

and the limiting value

$$\lim_{t \rightarrow \infty} \theta_1(t) = - [(\bar{\mu}_{11})']^{-1} \bar{\beta}_1 = \lambda_1.$$

In the special case in which $X^{[2]} \equiv 1$, the dynamics in (20)–(21) reduces to the lognormal model. The resulting elasticity is

$$\varepsilon(x, t) = \beta_d(x) + \alpha_d(x) \cdot [\bar{\alpha} + \bar{\sigma}_1' \theta_1(t)].$$

The term

$$\bar{\alpha} + (\bar{\sigma}_1)' (\exp [(\bar{\mu}_{11})'t] - I) [(\bar{\mu}_{11})']^{-1} \bar{\beta}_1$$

gives the vector of impulse responses of $\log M$ to the vector of Brownian increments. As is typical in the VAR literature with linear dynamics, the elasticity function is state-independent.

It is known from the VAR literature that the limiting large t response is the response of the martingale component of $\log M$ to the shock vector, which is given by

$$\bar{\alpha} + (\bar{\sigma}_1)' \lambda_1.$$

In our analysis we relate the limiting shock elasticity to the (proportionate) shock exposure of the martingale component of M , rather than of $\log M$. In the lognormal model these two entities coincide because the logarithm of the martingale component of M differs from the martingale component of the logarithm of M merely by a deterministic time trend. The time trend reflects the familiar lognormal adjustment for each horizon t . This simple connection between martingale components vanishes when we introduce nonlinearities in the drift coefficients and state dependence in the diffusion coefficients.

5.2 Long-run risk in consumption dynamics

We now add some economic structure to our previous example by exploring a “long-run risk” specification that has received recent prominence in the literature on asset pricing.¹⁶ This literature features models with a small predictable component in the growth rate of consumption and investors endowed with recursive utility preferences for which the intertemporal composition of risk matters. Stochastic volatility in the macroeconomy is included in

¹⁶For instance see Bansal and Yaron (2004).

part as a mechanism for risk prices to fluctuate over time.

Hansen et al. (2007) and Hansen (2009) present an example that is the continuous-time counterpart to the model of Bansal and Yaron (2004). This example utilizes the dynamic structure introduced in Section 5.1. In particular, consider an aggregate consumption functional C parameterized by (β_c, α_c) specified as in (21). A scalar process $X^{[1]}$ captures a statistically small but predictable component in the evolution of aggregate growth in consumption, and $X^{[2]}$ captures fluctuations in macroeconomic volatility. The Brownian motion is three-dimensional, and we will give an economic interpretation to the three shocks.

5.3 Investors' preferences

We compare the shock-price elasticities for two specifications of investors' preferences. In the Breeden (1979) and Lucas (1978) specification, investors have time-separable power utility with relative risk aversion coefficient γ . In the second specification, we endow investors with recursive preferences of the Kreps and Porteus (1978) or Epstein and Zin (1989) type, analyzed in continuous time by Duffie and Epstein (1992). We refer to the first model as the BL model and the second as the EZ model.

In the BL model, we immediately have the stochastic discount factor as:

$$S_t = \exp(-\delta t) \left(\frac{C_t}{C_0} \right)^{-\gamma}.$$

In the EZ model the stochastic discount factor requires more calculation. Let U denote the continuation value for the recursive utility specification and ϱ the inverse of the elasticity of intertemporal substitution. The continuous-time recursive utility evolution is restricted by:

$$0 = \frac{\delta}{1 - \varrho} [(C_t)^{1-\varrho} - (U_t)^{1-\varrho}] (U_t)^\varrho + \left[\frac{\Lambda_t}{(1 - \gamma)(U_t)^{1-\gamma}} \right] U_t$$

where Λ_t is the local mean:

$$\Lambda_t = \lim_{\epsilon \downarrow 0} \frac{E [(U_{t+\epsilon})^{1-\gamma} - (U_t)^{1-\gamma} | \mathcal{F}_t]}{\epsilon}.$$

Notice that this recursion is homogeneous of degree one in consumption and the continuation value process. The limiting version for $\varrho = 1$ is given as:

$$0 = \delta (\log C_t - \log U_t) U_t + \left[\frac{\Lambda_t}{(1 - \gamma)(U_t)^{1-\gamma}} \right] U_t. \quad (24)$$

In what follows we impose the unitary elasticity of substitution restriction as a device to

obtain a quasi-analytical solution for the continuation value. In this case, when $\log C_t$ is an additive functional of the Markov process translated by an initial contribution $\log C_0$, the continuation value satisfies:

$$\log U_t = \log C_t + v(X_t).$$

for some function v of the Markov state X_t . Specifically let

$$d \log C_t = \bar{\beta}_{c,0} dt + \bar{\beta}_{c,1} \cdot X_t^{[1]} dt + \bar{\beta}_{c,2} (X_t^{[2]} - 1) dt + \sqrt{X_t^{[2]}} \bar{\alpha}_c \cdot W_t$$

where the Markov process X is the one given in Section 5.1. Then it may be shown that (24) is equivalent to

$$\begin{aligned} 0 = & -\delta v(x) + \left[\frac{\partial v(x)}{\partial x} \right]' \begin{bmatrix} \bar{\mu}_{11} & \bar{\mu}_{12} \\ 0 & \bar{\mu}_{22} \end{bmatrix} \begin{bmatrix} x^{[1]} - \iota_1 \\ x^{[2]} - \iota_2 \end{bmatrix} + \bar{\beta}_{c,0} + \bar{\beta}_{c,1} \cdot (x^{[1]} - \iota_1) + \bar{\beta}_{c,2} (x^{[2]} - \iota_2) \\ & + \frac{x^{[2]}}{2} \text{trace} \left[\frac{\partial^2 v(x)}{\partial x \partial x'} \bar{\sigma} \bar{\sigma}' \right] + x^{[2]} (1 - \gamma) \left[\frac{\partial v(x)}{\partial x'} \right]' \bar{\sigma} \bar{\alpha}_c + \frac{x^{[2]} (1 - \gamma)}{2} |\bar{\alpha}_c|^2 \\ & + \frac{x^{[2]} (1 - \gamma)}{2} \left[\frac{\partial v(x)}{\partial x} \right]' \bar{\sigma} \bar{\sigma}' \left[\frac{\partial v(x)}{\partial x} \right]. \end{aligned} \tag{25}$$

This equation has a solution of the form:

$$v(x) = \bar{v}_0 + \bar{v}_1 \cdot x^{[1]} + \bar{v}_2 x^{[2]}.$$

Our primary interest is in the coefficients of the state vector x . From (25),

$$\begin{aligned} 0 = & -\delta \bar{v}_1 + (\bar{\mu}_{11})' \bar{v}_1 + \bar{\beta}_{c,1} \\ 0 = & -\delta \bar{v}_2 + (\bar{\mu}_{12})' \bar{v}_1 + (\bar{\mu}_{22}) \bar{v}_2 + \bar{\beta}_{c,2} + (1 - \gamma) (\bar{\alpha}_c)' (\bar{\sigma}_1)' \bar{v}_1 + (1 - \gamma) (\bar{\alpha}_c)' (\bar{\sigma}_2)' \bar{v}_2 \\ & + \frac{(1 - \gamma)}{2} \begin{bmatrix} (\bar{v}_1)' & \bar{v}_2 \end{bmatrix} \bar{\sigma} \bar{\sigma}' \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} + \frac{(1 - \gamma)}{2} |\bar{\alpha}_c|^2. \end{aligned}$$

We solve the first equation for \bar{v}_1 and substitute this solution into the second equation. The resulting equation is quadratic in \bar{v}_2 and typically has two solutions for appropriate choices of parameter values. Only one of these solutions interests us for the reasons discussed in Hansen and Scheinkman (2009a) and Hansen et al. (2007).

The resulting stochastic discount factor is:

$$S_t = \exp(-\delta t) \left(\frac{C_t}{C_0} \right)^{-1} \tilde{M}_t$$

where \tilde{M} is the multiplicative martingale from the shock exposure of the forward-looking logarithm of the continuation value function, $\log c + v(x)$, scaled by $1 - \gamma$. The martingale is

$$\log \tilde{M}_t = \int_0^t \sqrt{X_u^{[2]}} \tilde{\alpha} \cdot dW_u - \frac{|\tilde{\alpha}|^2}{2} \int_0^t X_u^{[2]} du \quad (26)$$

where

$$\tilde{\alpha} = (1 - \gamma) [(\bar{\sigma}_1)' \bar{v}_1 + (\bar{\sigma}_2)' \bar{v}_2 + \bar{\alpha}_c].$$

5.4 Elasticities

The BL and EZ structural specifications that we have just given imply a special case of the specification in Section 5.1. We use the formulas from that section to represent the elasticities that interest us. In the calculations that follow, we use parameter values from Hansen et al. (2007) that by design approximate the discrete-time specification in Bansal and Yaron (2004). The calculation is parameterized such that the innovations to $\log C$, $X^{[1]}$, and $X^{[2]}$ are mutually uncorrelated. We label these innovations as consumption, growth-rate, and volatility shocks, although a more fundamental structural model of the macroeconomy would, among other things, lead to more interesting labels assigned to shocks. We normalize the volatility shock so that a positive shock *reduces* volatility which is a good outcome, as are positive shocks to consumption and growth rates.

Recall that Section 3.2 presents one approach to computing price elasticities from component parts associated with the functionals G and $V = GS$. When the growth functional $G_t = \frac{C_t}{C_0}$, the product used for valuation is

$$V_t = G_t S_t = \exp(-\delta t) \tilde{M}_t,$$

where \tilde{M} is the forward-looking martingale component constructed from the value function in formula (26). The functional \tilde{M} is the martingale component of V , the eigenfunction $e \equiv 1$, and the eigenvalue is $-\delta$. As a result, the shock elasticity associated with V does not depend on the investment horizon and is given by:

$$\varepsilon_v(x, t) = x^{[2]} \tilde{\alpha} \cdot \bar{\alpha}_d$$

	$x_{25\%}^{[2]}$	$x_{50\%}^{[2]}$	$x_{75\%}^{[2]}$	$x^{[2]} = \iota_2$	$x^{[2]} = \tilde{\iota}_2$
consumption	0.20	0.24	0.28	0.24	0.27
growth-rate	0.38	0.45	0.53	0.46	0.51
volatility	0.09	0.10	0.12	0.10	0.11

Table 1: *Shock-value elasticities scaled by minus one*: The state dependence of this contribution is seen across columns, and the rows vary according to the shock chosen by α_d . Parameters are calibrated to monthly frequency, and the elasticities are annualized. The parameterization is $\bar{\beta}_{c,0} = 0.0015$, $\bar{\beta}_{c,1} = 1$, $\bar{\beta}_{c,2} = 0$, $\bar{\mu}_{11} = -0.021$, $\bar{\mu}_{12} = \bar{\mu}_{21} = 0$, $\bar{\mu}_{22} = -0.013$, $\bar{\alpha}_c = [0.0078 \ 0 \ 0]'$, $\bar{\sigma}_1 = [0 \ 0.00034 \ 0]$, $\bar{\sigma}_2 = [0 \ 0 \ -0.038]$, $\iota_1 = 0$, $\iota_2 = 1$, $\delta = 0.002$, $\gamma = 10$.

This elasticity is time-independent, but it does vary with the state $x^{[2]}$.

In what follows we consider three different specifications of $\bar{\alpha}_d$ given by the three coordinate vectors. Table 1 reports the valuation contribution to the shock-price elasticity for quartiles of the stationary distribution of $x^{[2]}$. We use the change of measure to analyze the shifted shock-price elasticities. The martingale \tilde{M} changes the implied stationary distribution for X , and in particular its distorted mean is

$$\tilde{\iota}_2 = \frac{\bar{\mu}_{22}}{\bar{\mu}_{22} + \bar{\sigma}_2 \tilde{\alpha}} \iota_2.$$

Since ε scales with $x^{[2]}$, this distorted mean scales the limiting elasticity function, which is given by

$$\tilde{\iota}_2 \tilde{\alpha} \cdot \bar{\alpha}_d.$$

Table 1 also reports this long-run distorted mean along with the mean under the original distribution. The shock-value elasticity for the exposure to uncertain macroeconomic growth is substantially larger than those for the other two elasticities. The elasticity for macroeconomic volatility uncertainty is the smallest and less than a third of that for growth uncertainty. The mean under the change of measure is close to the upper quartile of the original distribution. Thus the elasticities only show a modest increase relative to the original mean when we shift forward the exposure date as in (13) and (14).

Given that this construction implies time-independent valuation contributions, the sensitivity of the price elasticities to investment horizon comes from the exposure elasticities. These elasticities will depend on the investment horizon because consumption growth rates and volatility are predictable.

Figure 1 displays these shock-exposure elasticities. Notice that under this parameter configuration the shock-exposure elasticities are larger for exposure to growth rate risk than

volatility risk. The magnitudes are much smaller than the corresponding shock value elasticities reported in the $x^{[2]}$ column of Table 1. As a consequence, holding fixed the initial state, the risk-price elasticities will be flat with magnitudes very similar to those reported in the Table.¹⁷ The resulting shock-price elasticities for the EZ model are displayed in Figure 2. For comparison we also present the shock-price elasticities for the BL model. Since the investor preferences are time separable in the BL model, the shock-price elasticities have essentially the same shape as the shock-exposure elasticities reported in Figure 1. The scale difference between these elasticities reflects our setting of the risk aversion parameter $\gamma = 10$.¹⁸

Since the consumption shock has only a permanent impact on consumption, the associated shock-price elasticities coincide for the two utility specifications. In contrast, local elasticities for the growth-rate and volatility risk in the BL model are zero, while in the forward-looking EZ model the elasticities for arbitrarily short investment horizons remain bounded away from zero. In the EZ model, exposure of future consumption to growth-rate and volatility risk induces fluctuations in the continuation utility. As a consequence both the growth-rate state and volatility state evolution directly influence the equilibrium stochastic discount factor in the model with recursive utility investors. The corresponding shock-price elasticity function is close to flat for this model with the limit being similar to that for the BL model with the same value of γ for pricing exposure to the the growth-rate shock.¹⁹

Section 3.3 suggests a second approach for computing price elasticities. With this approach we extract the multiplicative martingale component of consumption to use as our growth functional G and build perturbations so that $GH(\mathbf{r})$ is also a multiplicative martingale. As a consequence the entire shock price elasticity comes from the study of $V = GS$ and the associated perturbations. The resulting functional V will not be a discounted multiplicative martingale and it will contribute to the shock-price dynamics. While $\beta_d = 0$ for the first approach, it is not zero for this second approach because the perturbations are no longer martingales. While conceptually different, this second construction yields in this case essentially the same numerical values for the shock-price elasticities.

To summarize, the recursive utility (EZ) specification of preferences with a unitary elas-

¹⁷The flat nature of risk-price elasticities for the EZ model in contrast to the BL model was featured in Hansen (2009). Hansen (2009) also studies the impact of changing the intertemporal elasticity of substitution away from unity. Our paper features shock-elasticities as building blocks for the risk-elasticities, but for simplicity considers only the case of an elasticity of intertemporal substitution equal to unity.

¹⁸We follow Bansal and Yaron (2004) in our choice of $\gamma = 10$. Some readers may be concerned with our large value of γ . Anderson et al. (2003) give a robust utility interpretation for the EZ model with a unitary elasticity of substitution for which the parameter γ reflects a concern for model specification instead of risk aversion. They also consider alternative approaches to calibration.

¹⁹Hansen (2009) provides results that link the limiting risk prices to the specification of the subjective rate of discount δ . As $\delta \searrow 0$ the limiting risk prices for the EZ specification and the BL specification converge to each other.

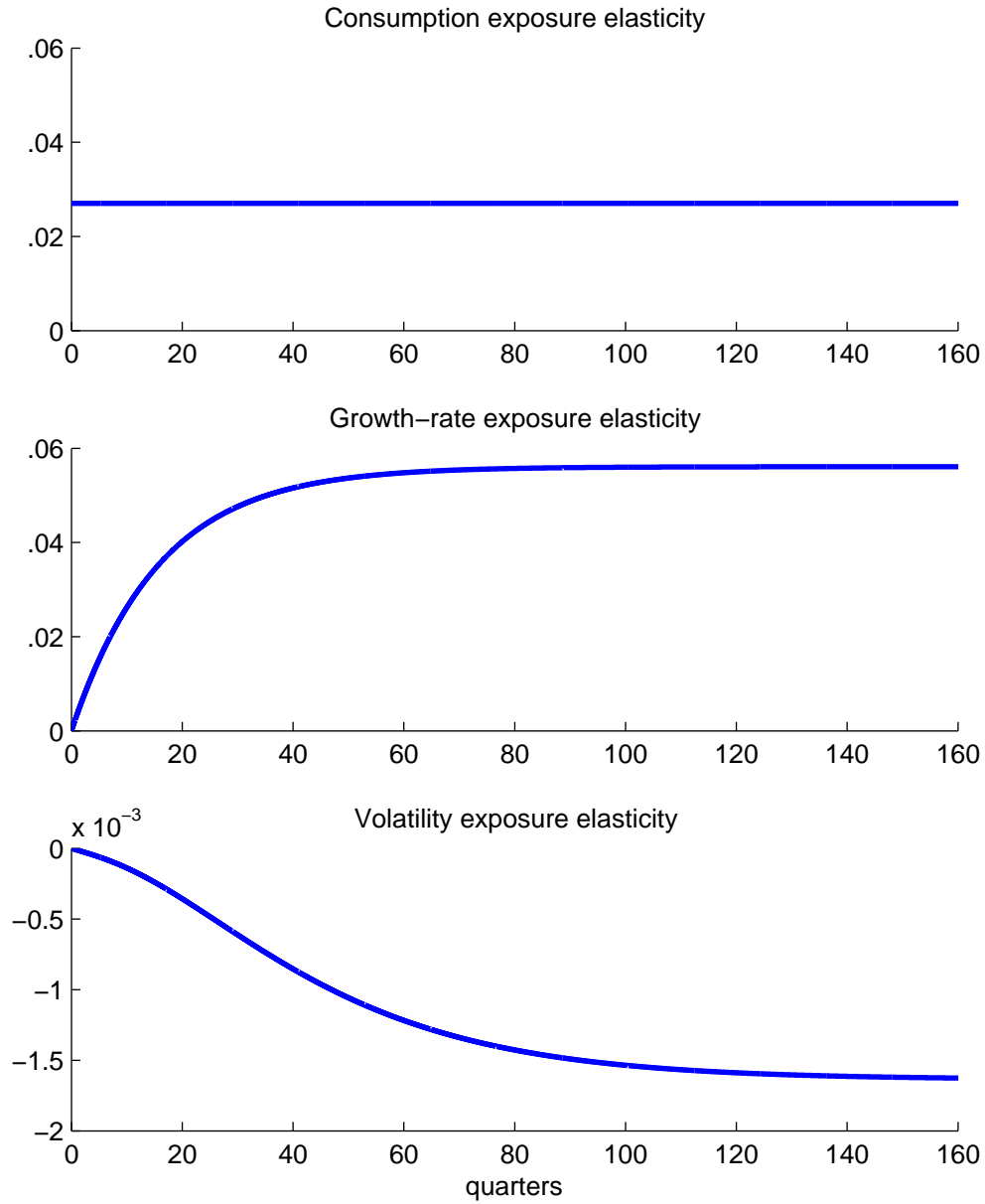


Figure 1: *Shock-exposure elasticities for the aggregate consumption process*: Elasticities for the three shocks are reported in the respective panels as functions of the forecast horizon. They condition on $x^{[2]} = \iota_2$. Parameter values are given in the caption to Table 1 and the elasticities are annualized.

ticity of intertemporal substitution and a risk parameter $\gamma = 10$ induces flat shock-price trajectories for growth rate and volatility shocks. This is in contrast to the power utility (BL) specification for which the shock-price elasticities are initially zero but become sizeable as we extend the investment horizon. In the BL model, the shock price elasticities are essentially scaled versions of the shock exposure elasticities for consumption.

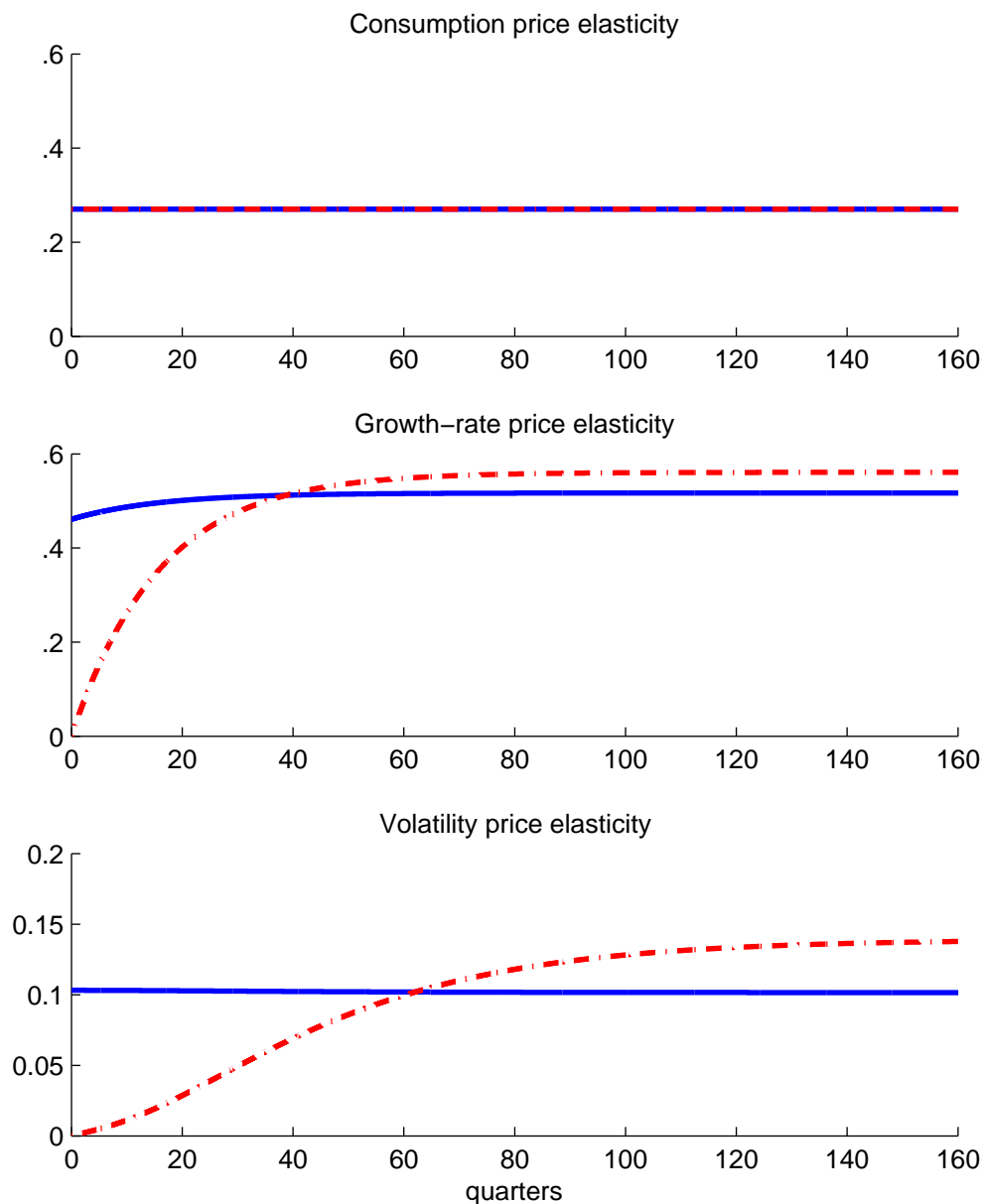


Figure 2: *Shock-price elasticities*: Elasticities for the three shocks are reported in the respective panels as functions of the investment horizon for the Breeden-Lucas (dashed line) and for the Epstein-Zin (solid line) preference specifications. They condition on $x^{[2]} = \iota_2$. Parameters are given in the caption to Table 1, and the elasticities are annualized.

6 Consumption externalities in investor preferences

In this section we contrast price elasticities of the example economies proposed by Campbell and Cochrane (1999) and Santos and Veronesi (2008). In these models, investor preferences include a prominent role for consumption externalities that are highly persistent.

6.1 A pricing example with nonlinearity

As a precursor to our study of models with consumption externalities in preferences, we consider an example with a nonlinearity in the Markov evolution. In this example the process for X is a member of Wong (1964)'s class of scalar Markov diffusions built to imply stationary densities that are in the Pearson family.²⁰

Let the univariate Markov state X evolve as:

$$dX_t = -\bar{\mu}_1(X_t - \bar{\mu}_2)dt - \bar{\sigma}X_t dW_t, \quad X_t > 0 \quad (27)$$

where $\bar{\mu}_1$, $\bar{\mu}_2$, and $\bar{\sigma}$ are positive constants.

Rather than specifying the multiplicative functional M and then calculating the factorization, we construct the multiplicative components directly as

$$\begin{aligned} M_t &= \exp(\eta t) \hat{M}_t \left(\frac{1 + X_t}{1 + X_0} \right) \\ \hat{M}_t &= \exp \left[-\frac{1}{2} \hat{\alpha}^2 t + \hat{\alpha} (W_t - W_0) \right] \end{aligned} \quad (28)$$

where $\hat{\alpha}$ is a constant. By construction, the dominant eigenfunction associated with M is $e(x) = (1 + x)^{-1}$ with eigenvalue η . Formula (8) then implies that the evolution of X under the change of measure is given by

$$dX_t = -\hat{\mu}_1 (X_t - \hat{\mu}_2) dt - \bar{\sigma} X_t d\hat{W}_t$$

where

$$\begin{aligned} \hat{\mu}_1 &= \bar{\mu}_1 + \bar{\sigma} \hat{\alpha} \\ \hat{\mu}_2 &= \frac{\bar{\mu}_1 \bar{\mu}_2}{\hat{\mu}_1}. \end{aligned}$$

Then $\hat{e}(x) = 1 + x$ and

$$\hat{E}[\hat{e}(x) \mid X_0 = x] = 1 + \hat{\mu}_2 + (x - \hat{\mu}_2) \exp(-\hat{\mu}_1 t).$$

As a consequence,

$$\phi(x, t) = -\bar{\sigma} x \frac{\exp(-\hat{\mu}_1 t)}{1 + \hat{\mu}_2 + \exp(-\hat{\mu}_1 t)(x - \hat{\mu}_2)}.$$

Since the exposure of $\log \hat{M}$ to the Brownian shock is the constant $\hat{\alpha}$, we can write the shock

²⁰See process F in Wong (1964).

elasticity (12) as

$$\begin{aligned}\varepsilon(x, t) &= \phi(x, t) + \hat{\alpha} + \beta_d(x) = \\ &= -\bar{\sigma}x \frac{\exp(-\hat{\mu}_1 t)}{1 + \hat{\mu}_2 + \exp(-\hat{\mu}_1 t)(x - \hat{\mu}_2)} + \hat{\alpha} + \beta_d(x)\end{aligned}$$

where we have set α_d to unity.

6.2 External habit model

The class of external habit models includes a variety of specifications that strive to explain empirical characteristics of the asset price dynamics. Important examples in this literature that we feature in our discussion are given in Campbell and Cochrane (1999) and in Santos and Veronesi (2008). As is well known in this literature, the local risk prices are systematically larger than the risk prices of a corresponding model with investors that have power utility preferences, and they vary over time even when consumption is a geometric Brownian motion.²¹ We will investigate the entire term structure of risk-price and shock-price elasticities as we extend the investment horizon. This is motivated in part by the observations in Santos and Veronesi (2008) and other papers about differences in returns on cash flows of alternative maturities. We compare these elasticities for the models of Campbell and Cochrane (1999) and Santos and Veronesi (2008) (abbreviated as CC and SV, respectively) and highlight important differences. In what follows, we start with the SV model which employs the dynamic structure we have introduced in Section 6.1 and for which there are closed-form solutions for the shock-price elasticities. For comparison we use a continuous-time version of the CC model and rely on numerical calculations similar to those in Wachter (2005).

Both models specify an aggregate consumption that evolves as a geometric Brownian motion:

$$d \log C_t = \bar{\beta}_c dt + \bar{\alpha}_c dW_t$$

where W is a scalar Brownian motion.

In addition both models specify the stochastic discount factor as a multiplicative func-

²¹We will hold the power used in depicting preferences the same across specifications but this comparison does not attempt to maintain the same degree of investor risk aversion.

tional

$$\begin{aligned} S_t &= \exp(-\delta t) \left(\frac{C_t - C_t^*}{C_0 - C_0^*} \right)^{-\gamma} \\ &= \exp(-\delta t) \left(\frac{C_t}{C_0} \right)^{-\gamma} \frac{\left(1 - \frac{C_0^*}{C_0} \right)^\gamma}{\left(1 - \frac{C_t^*}{C_t} \right)^\gamma}, \end{aligned}$$

where C^* is an external consumption reference (habit). SV and CC each chooses a different scalar state variable X_t that solves a stochastic differential equation on the Brownian motion W and where the dynamics guarantee that $C_t^* < C_t$.²² In what follows we will make comparisons between a model with investors that have preferences represented by discounted time-separable power utility (a Breeden-Lucas model in which $C^* = 0$) and a model in which there is a temporally dependent social externality in the stochastic discount factor for a decentralized economy. The reference to decentralization is important because internalizing the social externality would alter the stochastic discount factor. The growth functional G of primary interest to us is the aggregate consumption process itself:

$$G_t = \frac{C_t}{C_0}$$

for $t \geq 0$.

6.3 SV model specification

Santos and Veronesi (2008) choose a state variable

$$X_t = \left(1 - \frac{C_t^*}{C_t} \right)^{-\gamma} - 1$$

with the law of motion given in equation (27) in the previous Section 6.1. Since the process C_t is loglinear,

$$S_t = \exp(-\delta t) \left(\frac{C_t}{C_0} \right)^{-\gamma} \frac{e(X_0)}{e(X_t)}$$

where $e(x) = (1 + x)^{-1}$ is a principal eigenfunction. Then $M = SG$ is a multiplicative functional of the form given in (28), where $\hat{\alpha} = (1 - \gamma)\bar{\alpha}_c$. Additionally, the loading of X on

²²It is straightforward to allow the Brownian motion W to be multivariate. It could generate a larger filtration than the underlying Markov process X that will be introduced to model the consumption externality. What is critical is that the Markov dynamics are not altered with this more refined filtration. As emphasized to us by Eric Renault, this can be appropriately formulated as a statement that additional Brownian increments to be priced do not Granger-cause the underlying Markov process.

the shock, $\bar{\sigma}$, is expressed as a factor of $\bar{\alpha}_c$, $\bar{\sigma} = \chi \bar{\alpha}_c$.

Using the calculations from Section 6.1, the shock-price elasticity function is:

$$\varepsilon_p(x, t) = \chi \bar{\alpha}_c x \frac{\exp(-\hat{\mu}_1 t)}{1 + \hat{\mu}_2 + \exp(-\hat{\mu}_1 t)(x - \hat{\mu}_2)} + \gamma \bar{\alpha}_c$$

The local risk price (identical to the local shock-price elasticity) is

$$\varepsilon_p(x, 0) = \chi \bar{\alpha}_c \frac{x}{1 + x} + \gamma \bar{\alpha}_c$$

while the $t \rightarrow \infty$ limit is:

$$\varepsilon_p(x, \infty) = \gamma \bar{\alpha}_c.$$

This latter limit coincides with the shock-price elasticity function if consumption externality were absent from preferences ($e(x) = 1$). The impact of the consumption externality vanishes as the investment horizon increases, but this convergence will be slow when $\hat{\mu}_1$ is close to zero.

6.4 CC model specification

In Campbell and Cochrane (1999), the Markov state X is positive and solves

$$\left(1 - \frac{C_t^*}{C_t}\right)^\gamma = \exp[-\gamma(X_t + \mathbf{b})].$$

This state evolves as:

$$dX_t = -\xi(X_t - \mu_x) dt + \lambda(X_t) \bar{\alpha}_c dW_t \quad (29)$$

with the volatility factor $\lambda(x) = 1 - (1 + \zeta x)^{1/2}$ and $\zeta = 2\xi/(\gamma |\bar{\alpha}_c|^2)$. The functional form for the volatility factor is judiciously chosen to make the risk-free interest rate constant in much of their analysis. We provide details on the construction of the CC model in Appendix B. Hence again,

$$S_t = \exp(-\delta t) \left(\frac{C_t}{C_0}\right)^{-\gamma} \frac{e(X_0)}{e(X_t)},$$

where the principal eigenfunction e satisfies:

$$e(x) = \exp[-\gamma(x + \mathbf{b})]$$

This model specification implies the local shock-price elasticity

$$\varepsilon_p(x, 0) = \gamma\bar{\alpha}_c - \gamma\lambda(x)\bar{\alpha}_c = \gamma(1 + \zeta x)^{1/2}\bar{\alpha}_c,$$

and the $t \rightarrow \infty$ limit

$$\varepsilon_p(x, \infty) = \gamma\bar{\alpha}_c.$$

As in the SV specification, the CC model amplifies the local shock-price elasticities of the power utility model, $\gamma\bar{\alpha}_c$, by a state-dependent factor.

The shock- and risk-price elasticity functions must be computed numerically for the CC specification. The limiting elasticities as the payoff horizon $t \rightarrow \infty$ are the same for both specifications and coincide with those from a power utility model parameterized by the same γ . This simple conclusion masks some important differences, however. First we care about more than limits, but also about the speed of convergence. We will address this in detail in the subsequent discussion. Second, we have defined prices by taking derivatives, but there is an important discontinuity in the limiting risk premia in the CC specification, which we now investigate.

To see this, consider the limiting risk premium for the growth functional G :

$$\text{risk premium} = \lim_{t \rightarrow \infty} \frac{1}{t} (\log E[G_t | X_0 = x] - \log E[S_t G_t | X_0 = x] + \log E[S_t | X_0 = x])$$

where the last term is included to adjust for the long-term risk-free rate of interest. Consider a parameterized family of growth functionals $GH(r)$ built using volatility coefficients α of the form

$$\alpha_g(x) + r\alpha_d(x) = \bar{\alpha}_c + r$$

with corresponding drift coefficients β given by $-\frac{1}{2}(r + \bar{\alpha}_g)^2$. The limiting risk-price elasticity is the derivative of the risk premium with respect to r . In Appendix B we demonstrate that the long-term risk-price elasticity for the CC model is the same as that for a BL model with the same parameters except that $e = 1$.²³ Moreover, we show that the

$$\text{risk premium} = \gamma\bar{\alpha}_c(r + \bar{\alpha}_c) + \text{constant}, \tag{30}$$

²³We do not mean to imply that an econometrician or calibrator would select the same value of δ for each model. For instance, Campbell and Cochrane (1999) and Wachter (2005) use values of the subjective rate of discount that are much larger than would be used if the Breeden (1979) model was calibrated to asset return data. Even if the subjective rate of discount for the CC model is to fit interest rates, the calculation of r using this same subjective rate of discount, although counterfactual, is a revealing input into the risk-premia formula for the CC model.

where the size of the constant depends almost entirely on γ scaled by the predictability in consumption. Typically risk premia converge to zero as exposures converges to zero, but in the CC model there is a discontinuity in the behavior of the long-horizon limit. As $r + \bar{\alpha}_c$ converges to zero the limiting risk premium converges to a positive constant. At the parameter values suggested by Campbell and Cochrane (1999), this discontinuity is a sizable 7%. While this discontinuity is only present in the limit, it is indicative that risk price elasticities are large near $r + \bar{\alpha}_c = 0$ for valuation over long investment horizons. Campbell and Cochrane (1999) show that the conditional second moment of the stochastic discount factor diverges as the time horizon is extended. Our analysis gives a more refined characterization of the limiting behavior.²⁴

6.5 Model comparisons

To facilitate comparisons between the SV and CC specifications, we fix $\gamma = 2$ for both models but set the parameters of the SV model so that the distribution of local risk prices is similar to that in the CC model. Formally, the parameters μ_2 and χ are chosen to minimize the relative entropy of local risk price densities, which is the log-density ratio of local risk prices integrated with respect to the local risk price density of the SV model.²⁵ This is a convenient statistical measure of discrepancy constructed from stationary densities of risk prices.

Figure 3 reports the stationary densities for the local risk prices in the two models. In the top panel, the dashed curve shows why a recalibration of the SV model is needed to make meaningful comparisons. The original SV calibration leads to widely differing ranges for the local risk prices than those from the CC calibration. Even after adjusting the SV parameter values to try to make the local risk prices as similar as possible, the densities have rather different shapes. Using the same parameters γ and $\bar{\alpha}_c$ ensures that the long-horizon price elasticities coincide for the two models and equal to those for the counterpart BL model specification. Making both short-term and long-term elasticities similar allows us to focus at the relative differences in the pricing implications of the two models for finite-horizon cash flows.²⁶ In the comparisons that follow we set $\bar{\alpha}_d = 1$.

The top panel of Figure 4 displays the shock-price elasticity function for the quartiles

²⁴This discontinuity is absent in the SV specification and the long-term risk premia for the SV model agree with those of the corresponding BL model.

²⁵For the SV specification, it is tricky to change γ . If the specification of the consumption externality is held fixed the convenient functional form for the state evolution is lost.

²⁶While the long-term elasticities agree at positive values of $\bar{\alpha}_d$, as we have already argued the long-term risk premia are substantially different because of the discontinuity in the long-term risk premia in the CC model.

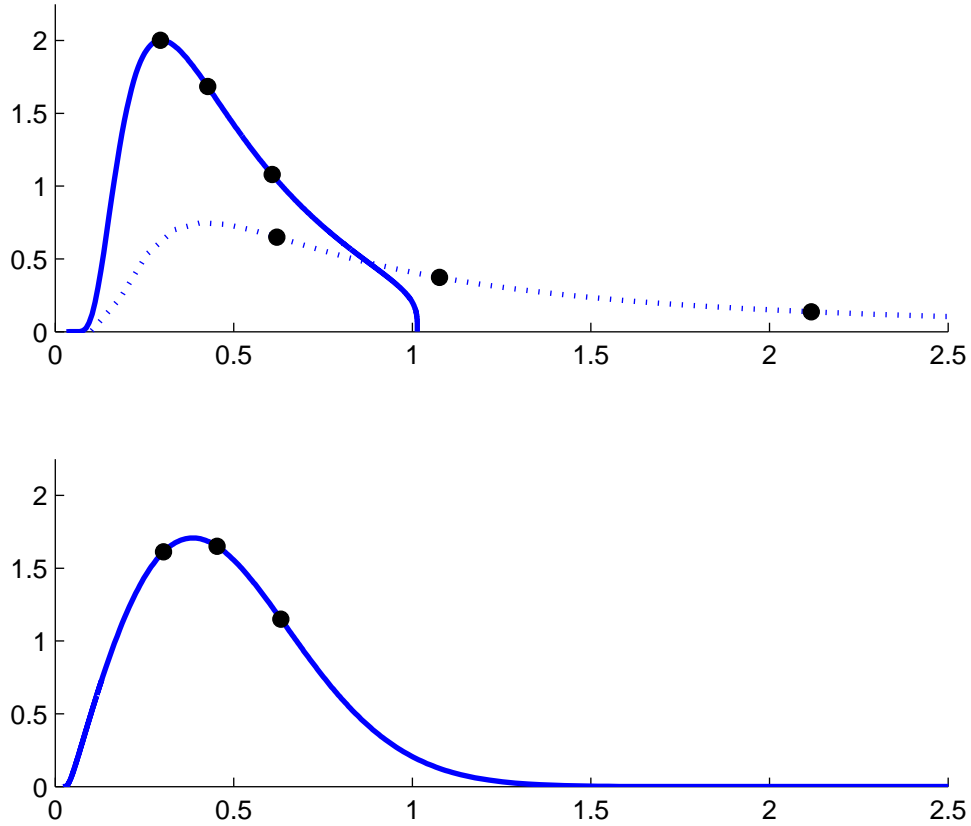


Figure 3: *Stationary densities for local risk prices*: The top panel displays the stationary density of local risk prices in the Santos and Veronesi (2008) model. The 25th, 50th, and 75th quantiles are marked with circles. The solid line represents our choice of parameters, $\chi = 91.9$, $\bar{\mu}_1 = 0.035$, $\bar{\mu}_2 = 2.335$, $\bar{\alpha}_c = 0.0054$, $\gamma = 2$. The dotted line shows the density for the original parameterization in the Santos and Veronesi (2008) model, $\chi = 538.6$, $\bar{\mu}_1 = 0.0325$, $\bar{\mu}_2 = 24.878$, $\bar{\alpha}_c = 0.0075$, $\gamma = 1.5$. The bottom panel compares with the model of Campbell and Cochrane (1999) with parameter values $\xi = 0.035$, $\mu_x = 0.4992$, $\bar{\alpha}_c = 0.0054$, and $\gamma = 2$. Parameters are calibrated to quarterly frequency, local risk prices annualized.

of the stationary distribution of the state variable X in the Santos and Veronesi (2008) model, and the bottom panel compares with the shock-price elasticity function implied by Campbell and Cochrane (1999). The elasticity function of the SV model decays relatively quickly and is near its limiting value by about 50 quarters. On the other hand, that of CC remains relatively flat for 100 quarters and does not approach its limiting value until about 300 quarters. Thus the SV model implies a much less persistent impact of exposure to a current shock on the prices of cash flows further in the future.

Recall that the shock-price elasticities depict the impact for valuation of shock exposure that occurs over the next instant. We now shift forward the date of the exposure to be τ

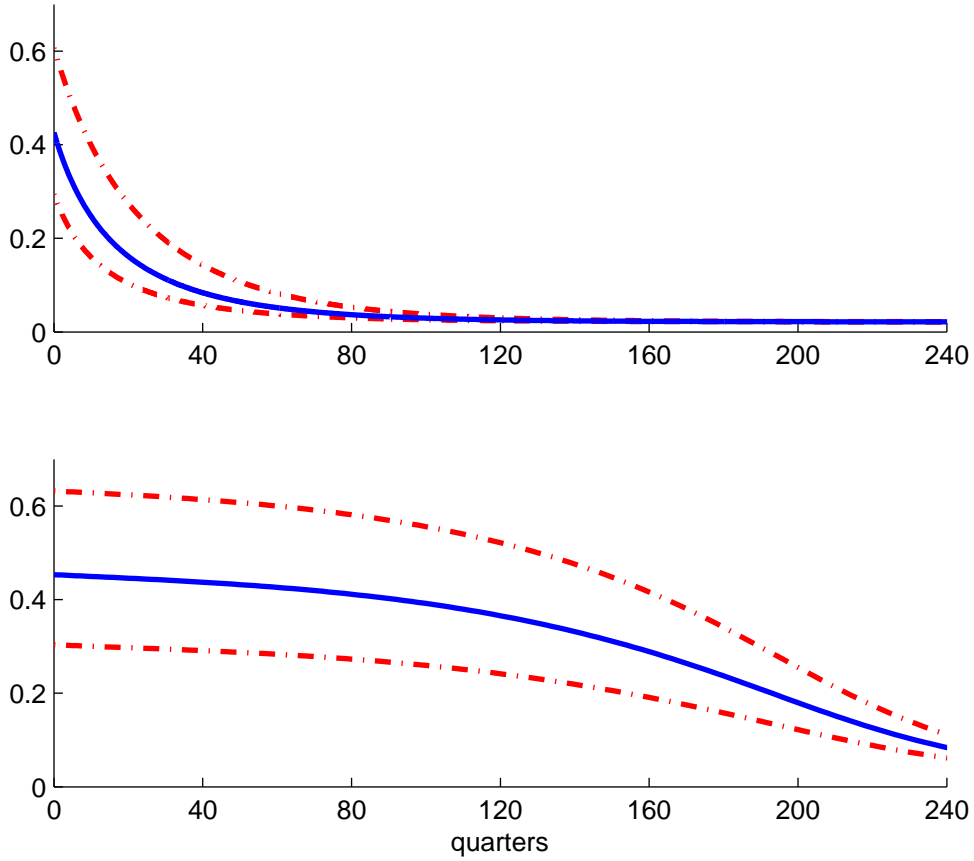


Figure 4: *Shock-price elasticities*: The top panel displays the shock-price elasticity function in the Santos and Veronesi (2008) model, while the bottom panel compares with the Campbell and Cochrane (1999) model. The solid curve conditions on the median state, while the dot-dashed curves condition on the 25th and 75th quantiles. The parameter values are given in the caption to Figure 3 and elasticities are annualized.

periods into the future. This gives the intermediate contributions to risk-price elasticities which are constructed in equation (13) as distorted conditional expectations of the shock-price elasticity function reported in Figure 4:

$$\varepsilon_p(x, t; \tau) = -\frac{\hat{E} [\hat{e}(X_{t+\tau}) [\alpha_s(X_\tau) + \phi(X_\tau, t) - \phi(X_\tau, 0)] | X_0 = x]}{\hat{E} [\hat{e}(X_{t+\tau}) | X_0 = x]}$$

where $t + \tau$ is the investment horizon. These curves (indexed by τ) have a well defined limit as $\tau \rightarrow \infty$ given by formula (14), which in the case of the SV model is

$$\varepsilon_p(x, t; \infty) = \gamma \bar{\alpha}_c + \exp(-\hat{\mu}_1 t) \frac{\hat{\mu}_2}{1 + \hat{\mu}_2} \chi \bar{\alpha}_c.$$

For the CC model counterpart we again rely on numerical calculations.

Figure 5 compares the limiting shock-price elasticities in the SV and CC models. Comparing the solid lines in the two panels, we see that in the SV model the limiting shock-price elasticities are much smaller than those of the CC model when $\bar{\alpha}_g = \bar{\alpha}_c$. While the SV limiting elasticities are moderately larger than their instantaneous counterparts (reported in Figure 4), the CC limiting elasticities are up to thirty times higher than their local counterparts. Evidently this property of the CC elasticities reflects a thick tail behavior of the distorted distributions for the shock elasticities.

The bottom panel of Figure 5 also depicts the limiting elasticities in the CC model for alternative exposures $\bar{\alpha}_g$. Previously we characterized a discontinuity in the long-term risk premia in the CC model at zero exposure. This plot sheds additional light on the source of this discontinuity. As we diminish the exposure level, the limiting elasticities become more substantial as might be expected. For small exposures to consumption risk in the distant future, the shock-price elasticities become huge in the CC model, especially those close to the payoff date. Since the shock-price elasticity $\varepsilon_p(x, t)$ converges to a constant $\gamma\bar{\alpha}_c$, so must the limiting contribution $\varepsilon_p(t; \infty)$ as $t \rightarrow \infty$, as long as $\bar{\alpha}_g > 0$. Thus the source of the discontinuity in the long-term risk premia is the exposure to shocks far in the future and close to the payoff date.²⁷

So far in this subsection, we have featured shock-price elasticities. We now consider the risk-price elasticities constructed by changing the exposure to risk to occur over the entire investment horizon. Recall that these risk-price elasticities are built up in Result 2.2 as integrals of distorted expectations of the shock-price elasticities over the lifetime of the cash flow. We depict the risk-price elasticities for the two models in Figure 6 as functions of the investment horizon. The top panel reports the risk-price elasticities in the SV model. At the median and higher quantiles of the state distribution, the risk-price elasticities *decrease* over all of the investment horizons. The decay rates for the risk-price elasticities are necessarily smaller than those for the shock-price elasticities because risk-price elasticities are averages of shock-price elasticities. In the CC model the risk-price elasticities increase with maturity up until about 200 quarters. This is consistent with the dramatic upward shift in the shock-price elasticities for the CC model as we move forward the exposure date as reflected in the bottom panel of Figure 5. It is only after 200 quarters that the risk-price elasticities begin to decrease in the investment horizon in the CC model, as we start averaging across

²⁷To elucidate the results reported calculation, consider the numerator of the limiting contribution in formula (14) for $t = 0$: $\hat{E} [\hat{e}(X_\tau)\varepsilon_p(X_\tau, 0)] = - \int \hat{q}(x)\hat{e}(x)\alpha(x)dx$ where $\hat{q}(x)$ denotes the stationary density for the state variable under the change of measure. A large x approximation of $\hat{q}(x)\hat{e}(x)$ is $\exp(-k\sqrt{x})$ while $-\alpha_d(x)$ behaves as \sqrt{x} for large x . In this approximation the coefficient $k \downarrow 0$ when $\bar{\alpha}_g \downarrow 0$, and as a consequence, $\hat{E} [\hat{e}(X_\tau)\varepsilon_p(X_\tau, 0)]$ diverges.

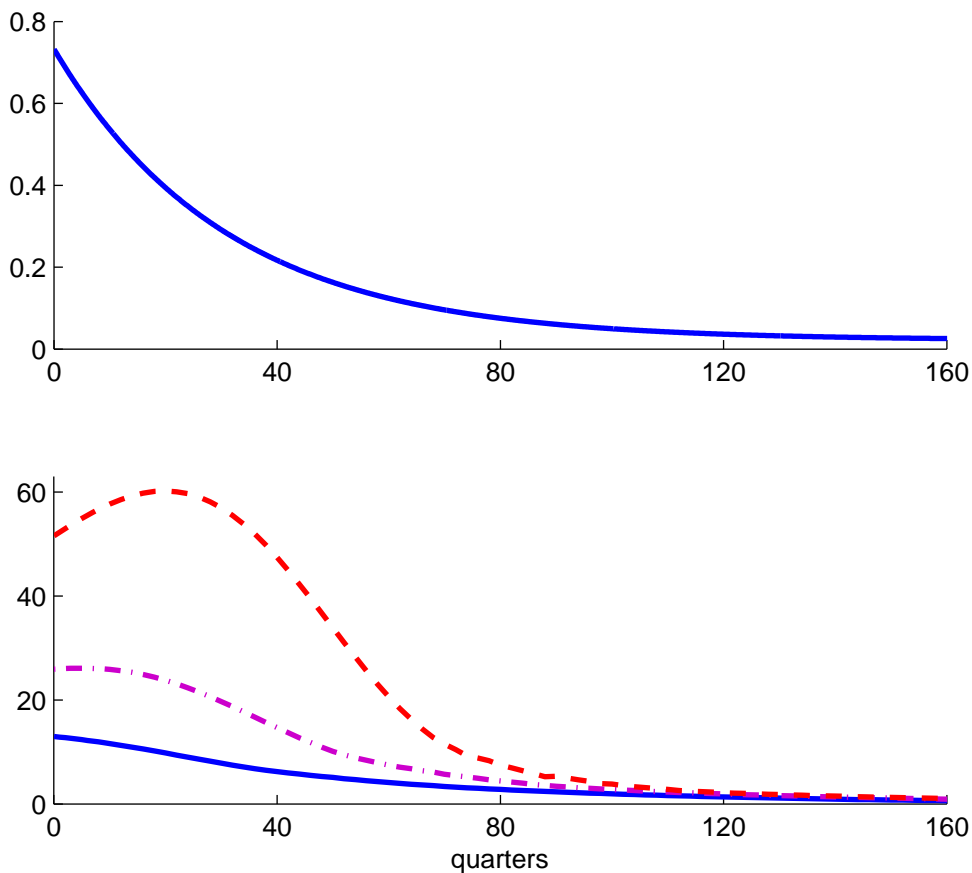


Figure 5: *Limiting shock-price elasticities*: The top panel displays the limiting shock-price elasticities for the Santos and Veronesi (2008) model. The bottom panel displays the limiting shock-price elasticity of the Campbell and Cochrane (1999) model for different levels of the baseline shock exposure $\bar{\alpha}_g$. The solid line represents $\bar{\alpha}_g = \bar{\alpha}_c$, the dot-dashed line $\bar{\alpha}_g = 0.5\bar{\alpha}_c$, and the dashed line $\bar{\alpha}_g = 0.25\bar{\alpha}_c$. The horizontal axis represents the distance between shock exposure and maturity of the cash flow. The parameter values are given in the caption to Figure 3 and elasticities are annualized.

the contributions of long-horizon shock-price elasticities that have small magnitudes (recall Figure 4).

In summary, the term structure of shock and risk-price elasticities are very different for the SV and CC models even though they were both designed to capture a similar empirical phenomenon, larger local risk prices in bad times than good times.

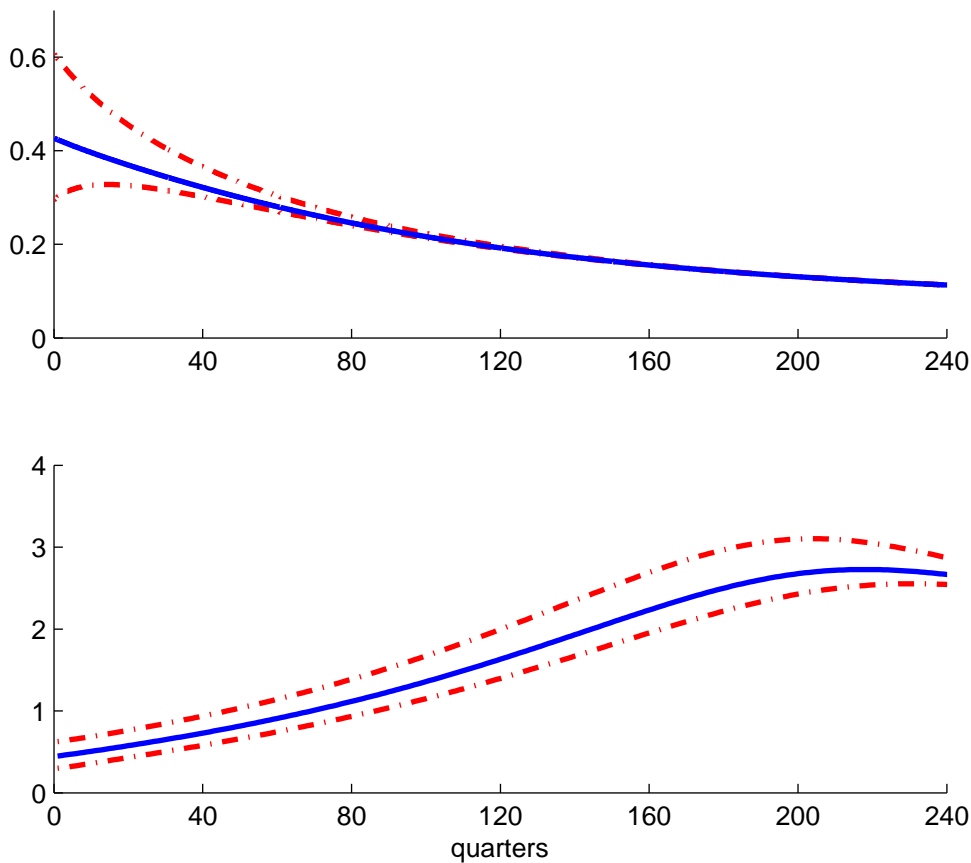


Figure 6: *Risk-price elasticities*: The top panel displays the risk-price elasticities as a function of investment horizon in the Santos and Veronesi (2008) model. The solid curve conditions on the median state, while the dot-dashed curves condition on the 25th and 75th quantiles. The bottom panel reports the counterpart plots for the Campbell and Cochrane (1999) model. The parameter values are given in the caption to Figure 3 and elasticities are annualized.

7 Incorporating jump risk

Thus far, we have analyzed models with Brownian information structure. In this section, we develop formulas that incorporate jumps in levels of the stochastic processes. We focus on a discrete state space specification with a finite number of states, where jumps are modeled as Poisson arrivals.²⁸ We use the jumps to mix alternative specifications or regimes, each of which is locally Gaussian and thus we explore pricing in a continuous-time version of the familiar regime shift model. We apply these results to price shocks associated with regime changes along with the exposure to Brownian motion. The remainder of this section is organized as follows. First we extend the construction of elasticities to accommodate a discrete-state Markov chain model of regimes, and then we illustrate these formulas using a

²⁸Bichteler et al. (1987) analyze jumps with Poisson arrivals on a continuous state space.

three-state model of consumption dynamics estimated by Bonomo and Garcia (1996).

7.1 Basics

Let W be a k dimensional Brownian motion and consider a functional M of the form

$$\log M_t = \sum_{0 < u \leq t} (Z_{u-})' \kappa Z_u + \int_0^t (Z_{u-})' \beta du + \int_0^t (Z_{u-})' \alpha dW_u. \quad (31)$$

Here Z evolves as an n -state Markov chain with intensity matrix A and the realizations of Z are identified by a coordinate vector in \mathbb{R}^n . We write Z_{t-} for the pre-jump (left) limit at date t . Abusing notation a bit, we now let β be an n -dimensional vector and α an $n \times k$ matrix. The functional is now parameterized by the triplet (β, α, κ) , representing the local mean conditional on no jumps, the local diffusive volatility, and the jumps in the functional. In this specification, the local trend and volatility depend (linearly) on the Markov state Z .

In our calculations in this section we use the following notational conventions. $\text{dvec}\{\cdot\}$ applied to a square matrix returns a column vector with entries given by the diagonal entries of the matrix, and $\text{diag}\{\cdot\}$ applied to a vector produces a diagonal matrix from a vector by placing entries of the vector on the corresponding diagonal entries of the constructed matrix. The symbol \times used in conjunction with two matrices forms a new matrix by performing multiplication entry by entry. $\exp^*(\cdot)$ when applied to a vector or matrix performs exponentiation entry by entry. Finally, a real-valued function on the state space of coordinate vectors can be represented as a vector.

7.2 Martingales

In this subsection we show how to construct multiplicative and additive martingale factorizations in the presence of jump components jump components.

7.2.1 Multiplicative martingales

We construct a multiplicative martingale factorization by computing an eigenfunction of the form $e \cdot z$ where the vector e has all positive entries. The vector e must solve the eigenvalue problem:²⁹

$$Be = \eta e \quad (32)$$

where

$$B \doteq \text{diag} \left\{ \beta + \frac{1}{2} \text{dvec} \{ \alpha \alpha' \} \right\} + A \times \exp^* (\kappa)$$

²⁹Details on the construction of the eigenvalue problems can be found in Appendix C.1.

Then

$$M_t = \exp(\eta t) \hat{M}_t \left(\frac{e \cdot Z_0}{e \cdot Z_t} \right) \quad (33)$$

and we can represent the martingale \hat{M} as

$$\log \hat{M}_t = \sum_{0 < u \leq t} (Z_{u-})' \hat{\kappa} Z_u + \int_0^t (Z_{u-})' \beta du + \int_0^t (Z_{u-})' \alpha dW_u - \eta t \quad (34)$$

where

$$\hat{\kappa} = \kappa + \mathbf{1}_n (\log e)' - (\log e) \mathbf{1}_n' \quad (35)$$

We use the multiplicative martingale \hat{M} to change the probability measure. This measure change leads to a Brownian motion \hat{W} under the new measure that satisfies

$$dW_t = (Z_{t-})' \alpha dt + d\hat{W}_t.$$

Under the new measure, the process Z has intensity matrix

$$\hat{A} = -\eta I + \text{diag}(\hat{e}) B \text{diag}(e)$$

where e and η are given by the solution of the eigenvalue problem (32), and \hat{e} is the vector of reciprocals of the entries in e .

7.2.2 Additive martingales

In order to construct perturbations corresponding to permanent shocks, we will extract the martingale component of an additive functional. Consider the martingale decomposition of the additive functional $\log M$ in (31)

$$\log M_t = \bar{\eta} t + \log \bar{M}_t - h \cdot Z_t + h \cdot Z_0. \quad (36)$$

To find the martingale component $\log \bar{M}$, let q denote a vector with positive entries that sum to one and satisfy

$$q' A = 0. \quad (37)$$

The long-run growth trend of the process is then given by

$$\bar{\eta} = q' \text{dvec} \{ \kappa A' \} + q' \beta. \quad (38)$$

The vector h determining the dominant component can be found as the solution to

$$Ah = -\text{dvec} \{ \kappa A' \} - \beta + \mathbf{1}_n \bar{\eta}. \quad (39)$$

Notice that the vector on the right-hand side is orthogonal to q , which is consistent with the fact that vectors in the image of A are orthogonal to q (see (37)). We solve equation (39) for h restricting ourselves to the $n - 1$ dimensional subspace of vectors that are orthogonal to q . The martingale component is then given by

$$\log \bar{M}_t = \sum_{0 < u \leq t} (Z_{u-})' \bar{\kappa} Z_u + \int_0^t (Z_{u-})' \bar{\beta} du + \int_0^t (Z_{u-})' \alpha dW_u \quad (40)$$

where

$$\begin{aligned} \bar{\beta} &= \beta - \mathbf{1}_n \bar{\eta} \\ \bar{\kappa} &= \kappa + \mathbf{1}_n h' - h \mathbf{1}'_n. \end{aligned}$$

Observe that $\bar{\kappa}$ has again zeros on the main diagonal. The permanent component of the jump risk is thus given by

$$(Z_{t-})' \bar{\kappa} Z_t + (Z_{t-})' \bar{\beta} dt.$$

We will also directly construct martingales. Consider an $n \times n$ matrix $\tilde{\kappa}$ with zeros on the diagonal, and build the additive martingale

$$\log \tilde{M}_t = \sum_{0 < u \leq t} (Z_{u-})' \tilde{\kappa} Z_u - \int_0^t (Z_{u-})' \text{dvec} \{ \tilde{\kappa} A' \} du.$$

For instance, we could specify all of the entries of $\tilde{\kappa}$ to be zero except for a single one.

Additive martingales scaled by the $\frac{1}{\sqrt{t}}$ obey the Central Limit Theorem. To deduce the variance ζ^2 associated with the normal approximation, the conditional (on $Z_{t-} = z$) second moment of the increment (per unit of time) is:³⁰

$$z' (\tilde{\kappa} \times \tilde{\kappa}) A' z.$$

Using the stationary distribution to average over alternative realizations of z ,

$$\zeta^2 = q \cdot \text{dvec} \{ (\tilde{\kappa} \times \tilde{\kappa}) A' \}.$$

³⁰Locally the second moment and variance coincide.

By scaling the matrix $\tilde{\kappa}$ by the scalar $\frac{1}{\zeta}$, we obtain an additive martingale with a unit unconditional variance per unit of time.

7.3 Jump-risk perturbations

Our jump-risk perturbations of a functional M are of the form $MH(\mathbf{r})$ where

$$\log H_t(\mathbf{r}) = \sum_{0 < u \leq t} (Z_{u-})' (\mathbf{r}\kappa_d) Z_u + \int_0^t (Z_{u-})' \beta_h(\mathbf{r}) du$$

where the direction matrix, κ_d , is the appropriately scaled (say $\frac{1}{\zeta}\tilde{\kappa}$) jump risk component in the direction of the desired perturbation, and $\beta_h(\mathbf{r})$ is a vector that induces $H(\mathbf{r})$ or $GH(\mathbf{r})$ to be a martingale, depending on the application. For the former, $\beta_h(\mathbf{r})$ needs to satisfy

$$0 = \beta_h(\mathbf{r}) + \text{dvec} \{ \exp^*(\mathbf{r}\kappa_d) A' \}.$$

Defining β_d analogously to the diffusion case, we have

$$\beta_d = \left. \frac{d\beta_h(\mathbf{r})}{d\mathbf{r}} \right|_{\mathbf{r}=0} = -\text{dvec} \{ \kappa_d A' \}$$

For the latter, recall that G is parameterized by $(\beta_g, \alpha_g, \kappa_g)$. Since the coefficients are additive, the appropriate martingale restriction determining $\beta_h(\mathbf{r})$ is

$$0 = \frac{1}{2} \text{dvec} \{ \alpha \alpha' \} + \beta + \beta_h(\mathbf{r}) + \text{dvec} \{ \exp^*(\kappa_g + \mathbf{r}\kappa_d) A' \}.$$

Differentiating with respect to \mathbf{r} and evaluating this derivative at zero, we have

$$\beta_d = -\text{dvec} \{ [\exp^*(\kappa_g) \times \kappa_d] A' \}.$$

7.4 Shock elasticities

We perform a direct calculation of the state-dependent moving-average coefficients needed to compute the elasticities that interest us. Recall that \hat{e} is the vector of reciprocals of the entries of e . Then

$$\begin{aligned} \hat{e} \cdot Z_t &= \sum_{0 < u \leq t} (Z_u - Z_{u-})' \exp \left[\hat{A}(t-u) \right] \hat{e} - \int_0^t (Z_{u-})' \hat{A} \exp \left[\hat{A}(t-u) \right] \hat{e} du \\ &\quad + (Z_0)' \exp \left(\hat{A}t \right) \hat{e} \end{aligned} \quad (41)$$

where \hat{A} is the intensity matrix under the change in measure. The new information at time u is

$$(Z_u - Z_{u-})' \exp \left[\hat{A}(t - u) \right] \hat{e} - (Z_{u-})' \hat{A} \exp \left[\hat{A}(t - u) \right] \hat{e} du,$$

and the first two terms in the decomposition (41) form a martingale.

Next we scale by $\zeta(t - u)' Z_{u-}$ where

$$\zeta(t - u) = \exp \left[\hat{A}(t - u) \right] \hat{e}$$

and produce a new representation that will be useful in our elasticity calculations. We do this in two steps:

1. First construct the matrix $\Xi(t - u)$ with (i, j) entry:

$$\Xi_{ij}(t - u) = \frac{\zeta_j(t - u)}{\zeta_i(t - u)} - 1$$

where $\zeta_i(t - u)$ is the i^{th} coordinate of $\zeta(t - u)$.

2. Second construct the vector $\xi(t - u)$ by dividing each entry of $\hat{A}\zeta(t - u)$ by the corresponding entry of $\zeta(t - u)$.

Then write

$$\begin{aligned} \hat{e} \cdot Z_t &= \sum_{0 < u \leq t} [\zeta(t - u)' Z_{u-}] [(Z_{u-})' \Xi(t - u) Z_u] - \\ &\quad - \int_0^t [\zeta(t - u)' Z_{u-}] [\xi(t - u)' Z_{u-}] du + \zeta(t)' Z_0. \end{aligned} \tag{42}$$

The moving-average representation for $\hat{e} \cdot Z_t$ derived in Section 7.4 allows us to state a counterpart of Result 2.2 for the jump risk case.

Result 7.1.

$$\begin{aligned} \frac{1}{t} \frac{d}{dr} \log E [M_t H_t(r) | Z_0 = z] \Big|_{r=0} &= \\ &= \frac{1}{t} \frac{\hat{E} \left[(\hat{e} \cdot Z_t) \int_0^t (Z_{u-})' \left(\beta_d + \text{dvec} \left\{ \kappa_d \hat{A}' \right\} + \text{dvec} \left\{ (\Xi(t - u) \times \kappa_d) \hat{A}' \right\} \right) du \mid Z_0 = z \right]}{\hat{E} [\hat{e} \cdot Z_t | Z_0 = z]} \end{aligned} \tag{43}$$

The proof is deferred to Appendix C.2. Switching the order of integration in the numerator of formula (43), the shock elasticity function for a direction κ_d is the time $u = 0$

contribution to the integral across the time dimension, viewed as a function of the maturity date t . We write the shock elasticity function as a vector

$$\varepsilon(t) = \beta_d + \text{dvec} \left\{ \kappa_d \hat{A}' \right\} + \text{dvec} \left\{ [\Xi(t) \times \kappa_d] \hat{A}' \right\}.$$

As in the Brownian case, we define the shock-price and shock-exposure elasticity functions by appropriately specifying the drift term β_d for suitable choices of the martingale M . In particular, we obtain the shock-price elasticity function as in Section 3.3 in the special case when $M = SG$ and both G and $GH(r)$ are martingales, attaching a minus sign by signing convention.

7.5 Growth and discounting

Our construction of the multiplicative functional M in (31) that explicitly allows for jumps in the levels of the functional is motivated by the implications of continuous-time Markov switching models for the equilibrium quantity dynamics that interest us. For the sake of illustration we introduce jumps directly in the growth or consumption processes, but production economies are also of interest and can be, and in fact have been, investigated using computational methods.

There are also other potential sources of jumps. We have already shown that even if a stochastic growth or discount functional contains no jumps in the sample paths ($\kappa = 0$), its martingale component both in additive and multiplicative form will generically contain a jump component provided there are jumps in either the conditional mean of the growth or conditional volatility. As we will see, when consumers have recursive preferences of the Epstein and Zin (1989) type the forward-looking continuation values may exhibit jumps even if the consumption process has a continuous sample path.³¹ Jumps thus become relevant when pricing permanent components of asset payoffs.

In some models with production and capital accumulation, jumps in the equilibrium consumption process may arise endogenously. In the Cagetti et al. (2002) model, the discrete Markov state determines the mean growth rate of the technology process, but the process itself has continuous trajectories. Since a regime shift discretely changes the instantaneous mean growth rate of the technology process and the conditional distribution of the future technology, there is also a discrete adjustment in the consumption and investment processes.³²

³¹Calvet and Fisher (2008), Chen (2008), and Bhamra et al. (2008) generate stochastic discount factors with discontinuous trajectories using the continuous-time version of Epstein and Zin (1989) preferences.

³²David (1997) produces a model along similar lines with two different linear technologies where jumps in the mean growth rates of the two technologies exactly offset each other, so that the distribution of

β_c	α_c	A		
0.0355	0.0330	-0.4627	0.4627	0
0.0127	0.0484	0.1709	-0.1938	0.0229
0.0193	0.0163	0.0554	0	-0.0554

Table 2: *Parameterization of the jump risk example*: The intensity matrix is calculated by taking the matrix logarithm of the transition probability matrix from Bonomo and Garcia (1996), and setting all negative off-diagonal terms equal to zero. This produces an intensity matrix with zeros in the same entries as in the original transition probability matrix. Original parameters estimated using yearly data from 1889–1985 (for details on the data sources, see Appendix A of the cited paper). Parameters are annualized.

In the following subsections, we construct the stochastic discount factor functional for the continuous-time version of the Epstein and Zin (1989) preferences when intertemporal elasticity of substitution is equal to one. We will subsequently use the stochastic discount factor to calculate the shock-price elasticities for consumption dynamics estimated by Bonomo and Garcia (1996).

7.6 Example economy with jumps

7.6.1 State dynamics

For illustrative purposes, we consider an example of consumption dynamics with three states, estimated by Bonomo and Garcia (1996). We focus on the pricing of permanent jump shocks to the equilibrium consumption stream in the Breeden-Lucas (BL) and Epstein-Zin (EZ) specification of preferences that we outlined in the example in Section 5.2.

Bonomo and Garcia (1996) specify the consumption dynamics as a conditionally Gaussian discrete-time process with jumps in the conditional growth rate and volatility. This leads us to parameterize consumption as a multiplicative functional given by $(\beta_c, \alpha_c, \kappa_c)$ where $\kappa_c = 0$ and scaled by the initial condition C_0 . Table 2 provides the parameter values β_c and α_c estimated by Bonomo and Garcia (1996). From filtered probabilities reported in that paper, we know that between 1890 and 1950 the economy spent most of the time switching between states 1 and 2, with longer spells spent in state 2.³³ Volatility is relatively high in both of these states. After 1950 the economy switched to the highly persistent, low-volatility, moderate mean-growth rate state 3, where it resides for most of the remainder of the sample (at least until recent events).

the aggregate production possibility set is independent of the current state. In this case, the equilibrium consumption process remains continuous.

³³See Figure 3 of Bonomo and Garcia (1996).

7.6.2 Investors' preferences

As in Section 5.2 we consider two models of preferences. Recall that in model BL the stochastic discount functional is:

$$S_t = \exp(-\delta t) \left(\frac{C_t}{C_0} \right)^{-\gamma}.$$

In model EZ we use the continuous-time specification of recursive utility preferences given in Section 5.2. As we saw, when preferences have a unitary elasticity the stochastic discount functional has a particularly simple structure:

$$S_t = \exp(-\delta t) \left(\frac{C_0}{C_t} \right) \tilde{M}_t$$

where \tilde{M} is the multiplicative martingale from the shock exposure of the logarithm of the forward-looking continuation value function scaled by $1 - \gamma$.

As in Hansen (2007), the logarithm of the equilibrium continuation value is of the form

$$\log V_t = v \cdot Z_t + \log C_t$$

where v solves the continuous-time discrete-state Bellman equation:

$$0 = -\delta v + \beta_c + \frac{1}{1 - \gamma} \text{dvec} \{ \exp^*[(1 - \gamma)\kappa_v] A' \} + \frac{1 - \gamma}{2} \text{dvec} \{ \alpha_c \alpha_c' \}. \quad (44)$$

and³⁴

$$\kappa_v = (\mathbf{1}_n v' - v \mathbf{1}_n') + \kappa_c.$$

To construct this martingale component, write

$$\begin{aligned} (1 - \gamma)(\log V_t - \log V_0) &= (1 - \gamma) [v \cdot (Z_t - Z_0) + \log C_t - \log C_0] \\ &= (1 - \gamma) \left[\sum_{0 < u \leq t} Z_{u-}' \kappa_v Z_u + \int_0^t Z_{u-}' \beta_c du + \int_0^t Z_{u-}' \alpha_c dW_u \right]. \end{aligned}$$

³⁴This equation is more general than the corresponding equation in Hansen (2007) because it allows for jumps in the consumption process and heteroskedasticity in the loading on the Brownian increment.

Then

$$\begin{aligned} \log \tilde{M}_t = & (1 - \gamma) \sum_{0 < u \leq t} Z_{u-}' \kappa_v Z_u - \int_0^t \text{dvec} \{ \exp^* [(1 - \gamma) \kappa_v] A' \} \cdot Z_{u-} du \\ & + (1 - \gamma) \int_0^t Z_{u-}' \alpha_c dW_u - \int_0^t \text{dvec} \left\{ \frac{(1 - \gamma)^2}{2} \alpha_c \alpha_c' \right\} \cdot Z_{u-} du. \end{aligned}$$

The coefficients in the stochastic discount functional thus are

$$\begin{aligned} \beta_s &= -\delta \mathbf{1}_n - \beta_c - \text{dvec} \{ \exp^* [(1 - \gamma) \kappa_v] A' \} - \text{dvec} \left\{ \frac{(1 - \gamma)^2}{2} \alpha_c \alpha_c' \right\} \\ &= -\delta \mathbf{1}_n - (1 - \gamma) \delta v - \gamma \beta_c \\ \alpha_s &= -\alpha_c + (1 - \gamma) \alpha_c = -\gamma \alpha_c \\ \kappa_s &= -\kappa_c + (1 - \gamma) \kappa_v = -\gamma \kappa_c + (1 - \gamma) (\mathbf{1}_n v' - v \mathbf{1}_n'). \end{aligned}$$

where we have used equation (44) for the vector δv .

7.6.3 Shock-price elasticities

We specify the growth functional G as the multiplicative martingale component of the consumption functional C (re-normalized conveniently to be one at date zero), extracted using the procedure outlined in Section 7.2.1. We illustrate our elasticity calculations by pricing the jump component of the permanent shock to $\log C$.³⁵ Let $\bar{\kappa}_c$ denote the corresponding jump matrix, which dictates how the shock is constructed as function of the jumps in Z . We parameterize the perturbation $H(\mathbf{r})$ using $(\beta_h(\mathbf{r}), 0, \mathbf{r} \bar{\kappa}_c)$ where $\beta_h(\mathbf{r})$ makes $GH(\mathbf{r})$ a martingale, and then we scale the perturbation by the reciprocal of the long-run volatility as in Section 7.2.2. This scaling normalizes the risk exposure of the shock. The resulting direction matrix is:

$$\frac{1}{\varsigma} \bar{\kappa}_c = \begin{pmatrix} 0 & -2.1804 & -0.1139 \\ 2.1804 & 0 & 2.0664 \\ 0.1139 & -2.0664 & 0 \end{pmatrix}$$

As reflected by the first row of this matrix, a movement from the first state to either of the other states has an adverse consequence on this permanent shock to consumption. In contrast movements from the second state to either of the first two states has a positive impact on the permanent shock. From the third state, a movement to the high-growth first

³⁵Using these same methods, elasticities can be computed for other shocks, including the Brownian increments or surprise movements from any state to another. A more comprehensive set of elasticities are reported in an earlier draft of this paper. See Borovička et al. (2009).

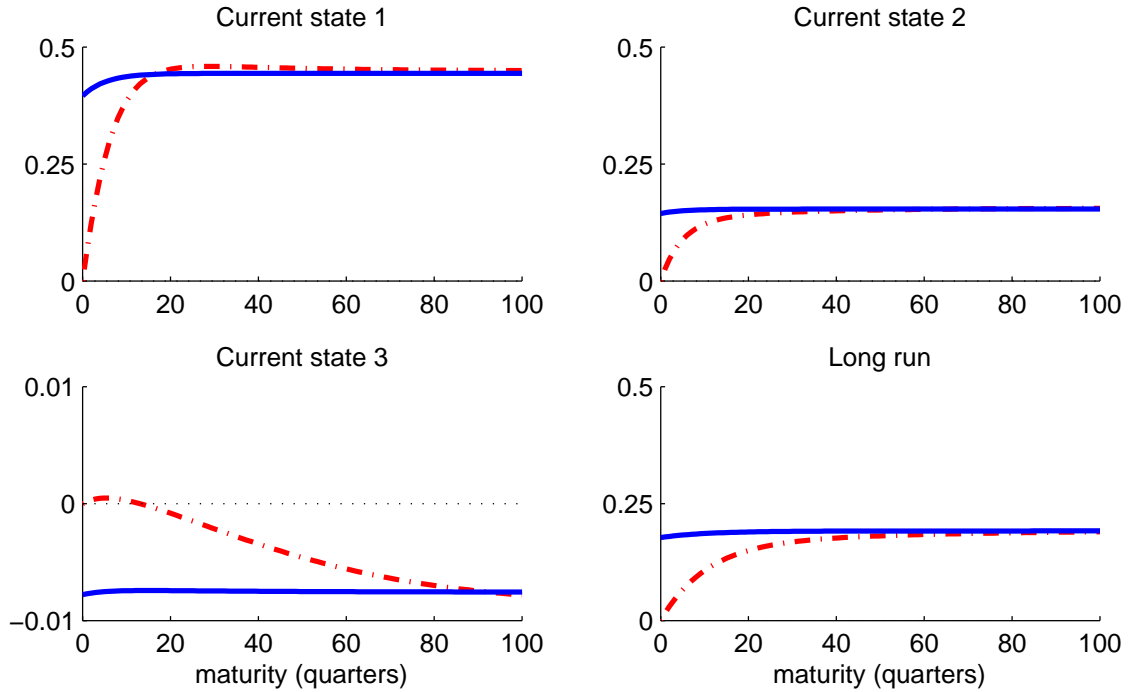


Figure 7: *Shock-price elasticities for the jump component of the permanent shock to consumption*: We constructed the permanent jump component using the Bonomo and Garcia (1996) consumption dynamics. The elasticities are depicted for the Breeden-Lucas (dot-dashed lines) and Epstein-Zin (solid lines) specifications of investor preferences. The growth functional is the martingale component in the multiplicative factorization of aggregate consumption, and the direction of the perturbation is given by the jump component of the martingale in the additive decomposition of $\log C$. Preference parameters are $\gamma = 10$ and $\delta = 0.01$, and elasticities are annualized.

state has a positive impact and to the low-growth second state a negative impact.

Figure 7 displays the shock-price elasticities for the two utility specifications conditioned on each of the three states as well as the limiting contribution (14).

The stochastic discount factor in the BL model has continuous sample paths. Since the diffusion and jump terms have independent increments, the local price elasticities with respect to the jump component are zero, as shown in Figure 7. The jump shock-price elasticities are not zero over time investment horizons, and as seen in the figure flatten out for horizons over five years. In the EZ specification, the elasticity trajectories are almost flat from the outset, reflecting the dominance in the martingale component of the stochastic discount factor. Also the price elasticity of this exposure to the permanent jump shock varies substantially depending upon the current state. While prices in state 1 are sizeable, the prices in state 3 are tiny. Bonomo and Garcia (1996) report filtered state probabilities

which indicate that the economy was switching between states 1 and 2 until about 1950, while in the post-1950 era the economy mostly resided in state 3. The limiting shock-price trajectory (14) is plotted in the bottom right panel of Figure 7. This curve shows that a forward shift in the time of exposure in the case of state 3 will substantially increase the shock-price elasticities while the same shift in case of state 1 leads to a drop by about fifty percent.

In contrast to the continuous-state specification studied in Section 5, the state dependence in this regime-shift model is substantial. In this discrete-state example the state evolution entails simultaneous changes in growth and volatility and is reflected in prices of exposure to the permanent shock to consumption. In the stochastic volatility model we studied, the permanent shock would be a combination of the direct shock to consumption and the shock to the growth rate of consumption.³⁶ Stochastic volatility scales this permanent shock but the shock to volatility does not contribute directly to the permanent shock. Thus the regime-shift model not only contributes fundamental discreteness through its use of only three realized states but it also features simultaneous changes in growth rates and volatility.

³⁶See Hansen (2009).

8 Conclusion

Stochastic dynamic model economies inform us how alternative shocks influence key economic variables at alternative time horizons. Structural models of asset valuation tell us even more. They inform us how the exposure to nondiversifiable macroeconomic shocks is compensated over alternative investment horizons. To understand better such implications, we proposed shock-price elasticities that measure this compensation and are valuation counterparts to impulse response functions. These price elasticities are also the dynamic extension of local risk prices familiar from finance by which exposure to shocks are assigned prices. Similarly, we constructed shock-exposure elasticities which capture the sensitivity of expected cash flows. We produced tractable continuous-time formulas for structural models that explicitly account for stochastic discounting and macroeconomic growth. Thus this paper provides an additional tool for analyzing structural models that connect macroeconomics and asset pricing.

In this paper we deduced price and exposure elasticities by *deconstructing* the risk premia of conveniently chosen cash flows. Risk premia on specific assets depend on the exposure of an underlying cash flow to risk along with the price of that exposure. By design our elasticity calculations explore marginal changes in exposures in alternative directions, and in models with nonlinearities these elasticities depend on what benchmark cash flow is used in their construction and on the evolution of the Markov state. *Constructing* risk premia thus requires that we integrate the marginal contributions over the range of the relevant exposures. This integration is implicit when we confront empirical evidence using a limited set of asset payoffs and prices. While we have not proposed a new set of statistical procedures for testing, we believe the deconstruction of risk premia to be of interest in understanding better the implications of alternative asset pricing models.

In a series of examples, we showed how to construct the shock elasticities in models where investors' preferences include recursive utility and external habit specifications, and where there is consumption predictability and stochastic volatility. We also explored models where the dynamics are driven, at least in part, by a finite-state Markov chain governing changes in regimes. We showed examples in which external habit models that have similar implications for local risk prices have dramatically different implications over long investment horizons, and examples of models with growth rate and volatility predictability (long-run risk models) which have similar long-term price implications but substantially different implications for shorter horizons.

While our examples feature alternative specifications of investor preferences, the starting point for the methods we develop is a benchmark macroeconomic growth process and a corresponding stochastic discount factor process. It is well known that models with explicit

investor heterogeneity in opportunities and limitations to the nature of asset trading can still be captured by appropriately specified stochastic discount factors. For instance, see Hansen and Renault (2009). We anticipate that a more comprehensive study of the pricing implications of these models will reveal interesting comparisons to some of the models that we have explored in this paper. While we have considered a jump risk models with finite states, we expect a richer investigation of Levy process within our framework to be a valuable extension.

Finally, we have abstracted from econometric and empirical challenges. While we leave this to future work, we do not wish to diminish the importance of these tasks. In regard to empirical implications, Bansal and Lehmann (1997), Alvarez and Jermann (2005), and Kojien et al. (2009) use the holding period return on long-term bonds and the maximal growth portfolio to gain information about the one-period stochastic discount factor in a discrete-time asset pricing model. The risk premium on the maximal growth portfolio reveals information on the volatility of the logarithm of the stochastic discount factor and the limiting holding-period return on a discount bond reveals the one-period ratio of the dominant eigenfunction ($e(X_{t+1})/e(X_t)$ in our notation) in a multiplicative factorization of the stochastic discount factor. They construct informative bounds on the logarithm of the stochastic discount factor and its components. While stochastic growth is not central to the valuation of fixed income securities, as we have seen the valuation of cash flows exposed to macroeconomic growth requires characterization of the co-dependence between stochastic discounting and stochastic growth. Thus empirical extensions of this literature that explicitly confront the valuation of stochastic growth are a potentially fruitful direction for future research. The initial steps by Lettau and Wachter (2007) and Hansen et al. (2008) are promising starts in this direction.

A Stochastic discount factor under recursive utility

We follow Duffie and Epstein (1992) in our construction of the stochastic discount factor.

- i) Take a monotone transformation of the utility index:

$$U_t^* = (U_t)^{1-\gamma}$$

For $\gamma > 1$, the case that interests us, this transformation is decreasing, so we will have to make an appropriate sign adjustment.

- ii) Notice that Λ is the local mean for U^* . Solve (24) for Λ :

$$\begin{aligned}\Lambda_t &= \delta(\gamma - 1) \left[\log C_t + \left(\frac{1}{\gamma - 1} \right) \log U_t^* \right] U_t^* \\ &\doteq \Psi(C_t, U_t^*).\end{aligned}$$

Let Ψ_i denote the partial derivative of Ψ with respect to its i^{th} argument. Compute:

$$\begin{aligned}\Psi_1(C_t, U_t^*) &= \delta(\gamma - 1) \left[\frac{U_t^*}{C_t} \right] \\ \Psi_2(C_t, U_t^*) &= \frac{\Lambda_t}{U_t^*} + \delta.\end{aligned}$$

- iii) Following Duffie and Epstein (1992) (see their formula (35)),

$$\begin{aligned}S_t &= \frac{\Psi_1(C_t, U_t^*)}{\Psi_1(C_0, U_0^*)} \exp \left[- \int_0^t \Psi_2(C_\tau, U_\tau^*) d\tau \right] \\ &= \exp(-\delta t) \left(\frac{C_0}{C_t} \right) \left(\frac{U_t^*}{U_0^*} \right) \exp \left[- \int_0^t \left(\frac{\Lambda_\tau}{U_\tau^*} \right) d\tau \right]\end{aligned}$$

where we placed a minus sign in front Ψ_2 to offset the fact that we used a monotone decreasing transformation of the utility index. Then the drift of the multiplicative functional $\left\{ \left(\frac{U_t^*}{U_0^*} \right) \exp \left[- \int_0^t \left(\frac{\Lambda_\tau}{U_\tau^*} \right) d\tau \right] : t \geq 0 \right\}$ is zero, and hence this process is an exponential local martingale, consistent with our conclusion in Section 5.2.

B The Campbell-Cochrane model

In this appendix we give some more details of the analysis of the Campbell-Cochrane model. Part of this discussion will be familiar to careful readers of Campbell and Cochrane (1999).

We include some repetition because we parameterize their model in a different (but equivalent) way. Also, in this appendix we allow for there to be a multivariate Brownian motion, although in the text we feature the case of scalar Brownian motion.

B.1 Risk-free rate

The instantaneous rate of interest for the Campbell and Cochrane (1999) model is:

$$-\lim_{t \downarrow 0} \frac{1}{t} \log E[S_t | X_0 = x] = \rho^* - \gamma \xi (x - \mu_x) - \gamma^2 \frac{\lambda(x)^2 |\bar{\alpha}_c|^2}{2} + \gamma^2 |\bar{\alpha}_c|^2 \lambda(x),$$

where ρ^* is the interest rate for the power utility specification in the absence of a consumption externality:

$$\rho^* = \delta + \gamma \bar{\beta}_c - \frac{\gamma^2 |\bar{\alpha}_c|^2}{2}.$$

Campbell and Cochrane suppose the risk-free rate is an affine function of the state: $\rho + \theta(x - \mu_x)$. Thus

$$\rho + \theta(x - \mu_x) = \rho^* + \gamma \xi (x - \mu_x) - \gamma^2 \frac{\lambda(x)^2 |\bar{\alpha}_c|^2}{2} + \gamma^2 |\bar{\alpha}_c|^2 \lambda(x). \quad (45)$$

We infer the value of ρ by setting $x = 0$:

$$\rho^* = \rho + (\theta - \gamma \xi) \mu_x$$

Substituting this formula into (45), by a simple complete-the-square argument:

$$(\theta - \gamma \xi)x - \frac{\gamma^2 |\bar{\alpha}_c|^2}{2} = -\frac{\gamma^2 |\bar{\alpha}_c|^2}{2} [\lambda(x) - 1]^2.$$

Thus

$$\begin{aligned} \lambda(x) &= 1 - (1 + \zeta x)^{1/2} \\ \zeta &\doteq \frac{2(\gamma \xi - \theta)}{\gamma^2 |\bar{\alpha}_c|^2} \end{aligned}$$

In the text we focused on the case in which $\theta = 0$, which is the same case that is featured in Campbell and Cochrane (1999). Wachter (2005) explores the more general case in her analysis of term structure implications.

B.2 Locally predictable consumption externality

Campbell and Cochrane (1999) propose that the risk exposure of C_t^* be zero when $X_t = \mu_x$. The idea is that C_t^* is locally predetermined. To understand the ramifications of this restriction, recall that

$$C_t^* = C_t - C_t \exp(-X_t - \mathbf{b}),$$

where we now will use the local predictability restriction to determine the coefficient \mathbf{b} . This coefficient \mathbf{b} is important in quantifying risk aversion. The standard measure of relative risk aversion is now state-dependent and given by

$$\text{risk aversion} = \gamma \exp(X_t + \mathbf{b}).$$

The local risk exposure for C_t^* is

$$C_t[1 - \exp(-X_t - \mathbf{b})]\bar{\alpha}_c \cdot dW_t + C_t \exp(-X_t - \mathbf{b})\lambda(X_t)\bar{\alpha}_c \cdot dW_t.$$

Thus we require that

$$1 + \exp(-x - \mathbf{b})[\lambda(x) - 1] = 0,$$

or

$$1 - \lambda(x) = \exp(x + \mathbf{b})$$

for $x = \mu_x$. Squaring the equation and multiplying by $\exp(-2\mu_x)$

$$\exp(-2\mu_x) \left(1 + \left[\frac{2(\gamma\xi - \theta)}{\gamma^2|\bar{\alpha}_c|^2} \right] \mu_x \right) = \exp(2\mathbf{b})$$

which determines \mathbf{b} . At this value of \mathbf{b} , the relative risk aversion measure is $\gamma[1 - \lambda(\mu_x)]$ when $x = \mu_x$.

As an extra parameter restriction, Campbell and Cochrane (1999) suggest requiring that the derivative of the risk exposure with respect to x be zero at μ_x :

$$\exp(-\mu_x - \mathbf{b})[1 - \lambda(\mu_x)] + \exp(-\mu_x - \mathbf{b})\lambda'(\mu_x) = 0,$$

or

$$\frac{1}{2} ([\lambda(\mu_x) - 1]^2)' = \lambda'(\mu_x)[\lambda(\mu_x) - 1] = [\lambda(\mu_x) - 1]^2.$$

Thus

$$\frac{\gamma\xi - \theta}{\gamma^2|\bar{\alpha}_c|^2} = 1 + \left[\frac{2(\gamma\xi - \theta)}{\gamma^2|\bar{\alpha}_c|^2} \right] \mu_x,$$

which is the restriction on the underlying parameters. Specifically,

$$\mu_x = \frac{1}{2} - \frac{\gamma^2 |\bar{\alpha}_c|^2}{2(\gamma\xi - \theta)}$$

Notice that we may now express λ as:

$$\begin{aligned} \lambda(x) - 1 &= - \left(1 + \left[\frac{2(\gamma\xi - \theta)}{\gamma^2 |\bar{\alpha}_c|^2} \right] x \right)^{1/2} \\ &= - \left(\frac{\gamma\xi - \theta}{\gamma^2 |\bar{\alpha}_c|^2} + \left[\frac{2(\gamma\xi - \theta)}{\gamma^2 |\bar{\alpha}_c|^2} \right] (x - \mu_x) \right)^{1/2} \\ &= - \left(\frac{\gamma\xi - \theta}{\gamma^2 |\bar{\alpha}_c|^2} \right)^{1/2} [1 + 2(x - \mu_x)]^{1/2}. \end{aligned}$$

as derived in Campbell and Cochrane (1999).

B.3 Change of measure

Parameterize the growth functional G by $(\bar{\beta}_g, \bar{\alpha}_g)$. The martingale component of SG is given by

$$\hat{M}_t = \exp \left[(\bar{\alpha}_g - \gamma\bar{\alpha}_c) \cdot (W_t - W_0) - \frac{t}{2} |\bar{\alpha}_g - \gamma\bar{\alpha}_c|^2 \right],$$

which we use to change the measure. With this change, the law of motion for X is:

$$dX_t = -\xi(X_t - \mu_x)dt + (\bar{\alpha}_g - \gamma\bar{\alpha}_c) \cdot \bar{\alpha}_c \lambda(X_t)dt + \lambda(X_t) \bar{\alpha}_c \cdot d\hat{W}_t. \quad (46)$$

Consider next the long-run behavior of value.

We use evolution equation (46), and the formula for the logarithmic derivative of the stationary density q for a scalar diffusion:

$$\frac{d \log q}{dx} = \frac{2 \text{ drift}}{\text{diffusion}} - \frac{d \log \text{diffusion}}{dx}$$

where the drift coefficient (local mean) is $-\xi(x - \mu_x)$ under the original measure or

$$-\xi(x - \mu_x) + (\bar{\alpha}_g - \gamma\bar{\alpha}_c) \cdot \bar{\alpha}_c \lambda(X_t)$$

under the twisted measure. The diffusion coefficient (local variance) is $\lambda(x)^2 |\bar{\alpha}_c|^2$.

The limiting behavior is dominated by the constant term:

$$\lim_{x \rightarrow \infty} \frac{d \log q}{dx} = -\frac{\gamma^2 \xi}{\gamma\xi - \theta} < 0. \quad (47)$$

As a consequence the process X is stationary under the original and under the twisted probability measure as reflected by (29) and (46) respectively. It remains to study what functions have finite moments under the twisted evolution.

When $\gamma\xi > \theta > 0$, $\exp(\gamma X_t)$ has a finite expectation under the twisted stationary density because the limit in (47) is strictly less than $-\gamma$. In contrast, when $\theta < 0$ this expectation will be infinite. Thus when $\theta > 0$ the contribution to preferences will be transient, but not when $\theta < 0$.

When $\theta = 0$, a more refined calculation is required because $\log q$ behaves like a positive scalar multiple of $-\gamma x$ for large x . This motivates the following:

$$\lim_{x \rightarrow \infty} \sqrt{x} \left(\frac{d \log q}{dx} + \gamma \right) = -2 \left(\frac{\bar{\alpha}_g \cdot \bar{\alpha}_c}{\bar{\alpha}_c \cdot \bar{\alpha}_c} \right) \zeta^{-1/2} = -\frac{\bar{\alpha}_g \cdot \bar{\alpha}_c}{|\bar{\alpha}_c|} \sqrt{\frac{2\gamma}{\xi}}.$$

For the contribution of e to the stochastic discount factor to have transient implications for valuation, this term must be negative because twice this limit is the coefficient on \sqrt{x} in the large x approximation of $\log q(x) + \gamma x$. While this term is zero when $\bar{\alpha}_g$ is zero, it will be negative provided that the shocks to $\log G_t$ and $\log C_t$ are positively correlated.

We now characterize the limiting risk premia. When $\theta > 0$,

$$\mathbf{risk\ premium} = \gamma \bar{\alpha}_c \cdot \bar{\alpha}_g$$

as in the Breeden (1979) model. When $\theta = 0$ and $\bar{\alpha}_c \cdot \bar{\alpha}_g > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log E [S_t G_t | X_0 = x] = \mu_g - \delta - \gamma \bar{\beta}_c - \gamma \bar{\alpha}_c \cdot \bar{\alpha}_g + \frac{\gamma^2}{2} |\bar{\alpha}_c|^2 + \frac{|\bar{\alpha}_g|^2}{2},$$

but

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log E [S_t | X_0 = x] = -\rho$$

This justifies formula (30).

C Derivations and proofs for Section 7

C.1 Eigenvalue problems

For the multiplicative decomposition in Section 7.2.1, guess that the martingale component takes the form (34). The martingale restriction for an increment in \hat{M} conditional on state $Z_{t-} = z_i$ is

$$0 = \frac{1}{2} z_i' \alpha \alpha' z_i + \beta_i - \eta + \sum_j A_{ij} \exp(\hat{\kappa}_{ij})$$

Plugging this restriction into decomposition (33) and comparing coefficients, we obtain the condition

$$\hat{\kappa}_{ij} = \log e_j - \log e_i + \kappa_{ij}$$

which yields equation (35) by stacking. Using this condition in the martingale restriction leads to

$$0 = \frac{1}{2} z'_i \alpha \alpha' z_i + \beta_i - \eta + \sum_j A_{ij} \frac{e_j}{e_i} \exp(\kappa_{ij})$$

which, after multiplying by e_i and stacking the equations, yields the eigenvalue equation (32).

The additive decomposition in Section 7.2.2 is obtained in a similar way. Guess the form of the martingale component $\log \bar{M}$ given by equation (40). The additive martingale restriction conditional on state $Z_{t-} = z_i$ is

$$0 = \sum_j \bar{\kappa}_{ij} A_{ij} + \bar{\beta}_i$$

Using this restriction in (36), and comparing coefficients, we have

$$\begin{aligned} \bar{\beta}_i &= \beta_i - \bar{\eta} \\ \bar{\kappa}_{ij} &= \kappa_{ij} + h_j - h_i \end{aligned}$$

Thus the martingale restriction implies

$$\bar{\eta} = \sum_j \kappa_{ij} A_{ij} + \sum_j A_{ij} h_j + \beta_i \quad (48)$$

Stacking this set of equations and premultiplying by q yields equation (38). The vector hh representing the dominant component can then be found (up to scale) as a solution to the system of equations (48).

C.2 Proof of Proposition 7.1

Proof. Notice that

$$\frac{d}{dr} \log E [M_t H_t(r) \mid Z_0 = z] \Big|_{r=0} = \frac{\hat{E} [(\hat{e} \cdot Z_t) D_t \mid Z_0 = z]}{\hat{E} [\hat{e} \cdot Z_t \mid Z_0 = z]}$$

where

$$D_t = \sum_{0 < u \leq t} (Z_{u-})' \kappa_d Z_u + \int_0^t (Z_{u-})' \beta_d du.$$

Further observe that the additive functional

$$\bar{D}_t = \sum_{0 < u \leq t} (Z_{u-})' \kappa_d Z_u - \int_0^t (Z_{u-})' \text{dvec} \left\{ \kappa_d \hat{A}' \right\} du$$

is a martingale under the change of probability measure. In order to find the expression for $\hat{E} [(\hat{e} \cdot Z_t) \bar{D}_t \mid Z_0 = z]$, we calculate the local covariance between corresponding increments in \bar{D} and the moving-average decomposition of $\hat{e} \cdot Z_t$ in formula (42). We have

$$\begin{aligned} \hat{E} [(\hat{e} \cdot Z_t) \bar{D}_t \mid Z_0 = z] &= \\ &= \hat{E} \left[\int_0^t [\zeta(t-u)' Z_{u-}] \left[(Z_{u-})' \text{dvec} \left\{ (\Xi(t-u) \times \kappa_d) \hat{A}' \right\} \right] du \mid Z_0 = z \right] \\ &= \hat{E} \left[\int_0^t (\hat{e} \cdot Z_t) \left[(Z_{u-})' \text{dvec} \left\{ (\Xi(t-u) \times \kappa_d) \hat{A}' \right\} \right] du \mid Z_0 = z \right] \end{aligned}$$

where we used

$$z' [(\Xi(t-u) \times \kappa_d) \hat{A}'] z = z' \text{dvec} \left\{ [\Xi(t) \times \kappa_d] \hat{A}' \right\}$$

and

$$[\zeta(t-u)' Z_{u-}] = \hat{E} [\hat{e} \cdot Z_t \mid Z_{u-}].$$

Combining this result with the expression for $\hat{E} [(\hat{e} \cdot Z_t) (D_t - \bar{D}_t) \mid Z_0 = z]$ completes the proof. \square

References

- Alvarez, Fernando and Urban J. Jermann. 2005. Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth. *Econometrica* 73 (6):1977–2016.
- Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent. 2003. A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection. *Journal of the European Economic Association* 1:69–123.
- Bansal, Ravi and Bruce N. Lehmann. 1997. Growth-Optimal Portfolio Restrictions on Asset Pricing Models. *Macroeconomic Dynamics* 1:333–354.
- Bansal, Ravi and Amir Yaron. 2004. Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles. *The Journal of Finance* 59 (4):1481–1509.
- Bhamra, Harjoat S., Lars-Alexander Kuehn, and Ilya A. Strebulaev. 2008. The Levered Equity Risk Premium and Credit Spreads: A Unified Framework.
- Bichteler, Klaus, Jean-Bernard Gravereaux, and Jean Jacod. 1987. *Malliavin Calculus for Processes with Jumps*. Gordon and Breach Science Publishers.
- Bismut, Jean-Michel. 1981. Martingales, the Malliavin Calculus and Hypocoellipticity under the General Hörmander’s Conditions. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 56:469–505.
- Bonomo, Marco and René Garcia. 1996. Consumption and Equilibrium Asset Pricing: An Empirical Assessment. *Journal of Empirical Finance* 3 (3):239–265.
- Borovička, Jaroslav, Lars Peter Hansen, Mark Hendricks, and José A. Scheinkman. 2009. Risk Price Dynamics. NBER Working Paper 15506.
- Breedon, Douglas T. 1979. An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities. *Journal of Financial Economics* 7 (3):265–296.
- Cagetti, Marco, Lars Peter Hansen, Thomas J. Sargent, and Noah Williams. 2002. Robustness and Pricing with Uncertain Growth. *The Review of Financial Studies* 15 (2):363–404.
- Calvet, Laurent E. and Adlai J. Fisher. 2008. Multifrequency Jump-Diffusions: An Equilibrium Approach. *Journal of Mathematical Economics* 44 (2):207–226.
- Campbell, John Y. and John Cochrane. 1999. By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior. *Journal of Political Economy* 107 (2):205–251.

- Chen, Hui. 2008. Macroeconomic Conditions and the Puzzles of Credit Spreads and Capital Structure. Forthcoming in *Journal of Finance*.
- David, Alexander. 1997. Fluctuating Confidence in Stock Markets: Implications for Returns and Volatility. *The Journal of Financial and Quantitative Analysis* 32 (4):427–462.
- Detemple, Jerome B. and Fernando Zapatero. 1991. Asset Prices in an Exchange Economy with Habit Formation. *Econometrica* 59 (6):1633–1657.
- Duffie, Darrell and Larry G. Epstein. 1992. Asset Pricing with Stochastic Differential Utility. *The Review of Financial Studies* 5 (3):411–436.
- Duffie, Darrell and Rui Kan. 1994. Multi-Factor Term Structure Models. *Philosophical Transactions: Physical Sciences and Engineering* 347 (1684):577–586.
- Epstein, Larry G. and Stanley E. Zin. 1989. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* 57 (4):937–969.
- Fournié, Eric, Jean-Michel Lasry, Jérôme Lebuchoux, and Pierre-Louis Lions. 1999. Applications of Malliavin Calculus to Monte-Carlo Methods in Finance. *Finance and Stochastics* 3 (4):391–412.
- Gourieroux, C. and J. Jasiak. 2005. Nonlinear Innovations and Impulse Responses with Application to VaR Sensitivity. *Annales d’Economie et de Statistique* 78:1–33.
- Grasselli, Martino and Claudio Tebaldi. 2004. Bond Price and Impulse Response Function for the Balduzzi, Das, Foresi and Sundaram (1996) Model. *Economic Notes* 33 (3):359–374.
- Hansen, Lars Peter. 2007. Beliefs, Doubts and Learning: Valuing Macroeconomic Risk. *American Economic Review* 97:1–30.
- . 2009. Modeling the Long Run: Valuation in Dynamic Stochastic Economies. Fisher-Schultz Lecture at the European Meetings of the Econometric Society.
- Hansen, Lars Peter and Eric Renault. 2009. Pricing Kernels. Forthcoming in *Encyclopedia of Quantitative Finance*.
- Hansen, Lars Peter and José A. Scheinkman. 2009a. Long-Term Risk: An Operator Approach. *Econometrica* 77 (1):177–234.
- . 2009b. Pricing Growth-Rate Risk. Forthcoming in *Finance and Stochastics*.

- Hansen, Lars Peter, John C. Heaton, Junghoon Lee, and Nikolai Roussanov. 2007. Intertemporal Substitution and Risk Aversion. In *Handbook of Econometrics: Volume 6A*, chap. 61, 3967–4056. Elsevier.
- Hansen, Lars Peter, John C. Heaton, and Nan Li. 2008. Consumption Strikes Back?: Measuring Long-Run Risk. *Journal of Political Economy* 116:260–302.
- Hausmann, Ulrich G. 1979. On the Integral Representation of Functionals of Ito Processes. *Stochastics and Stochastics Reports* 3 (1):17–27.
- Ito, Kiyosi and Shin Chi Watanabe. 1965. Transformation of Markov Processes by Multiplicative Functionals. *Annals of Institute Fourier* 15:13–30.
- Koijen, Ralph S.J., Hanno Lustig, and Stijn van Nieuwerburgh. 2009. The Cross-Section and Time-Series of Stock and Bond Returns.
- Kreps, David M. and Evan L. Porteus. 1978. Temporal Resolution of Uncertainty and Dynamic Choice Theory. *Econometrica* 46 (1):185–200.
- Len, Jorge A., Rayla Navarro, and David Nualart. 2003. An Anticipating Calculus Approach to the Utility Maximization of an Insider. *Mathematical Finance* 13 (1):171–185.
- Lettau, Martin and Jessica Wachter. 2007. Why is Long-Horizon Equity Less Risky? A Duration-Based Explanation of the Value Premium. *Journal of Finance* 62:55–92.
- Lucas, Jr., Robert E. 1978. Asset Prices in an Exchange Economy. *Econometrica* 46 (6):1429–1445.
- Ocone, Daniel L. and Ioannis Karatzas. 1991. A Generalized Clark Representation Formula, with Applications to Optimal Portfolios. *Stochastics* 34 (3):187–220.
- Øksendal, Bernt. 1997. An Introduction to Malliavin Calculus with Applications to Economics. Unpublished manuscript.
- Revuz, Daniel and Marc Yor. 1991. *Continuous Martingales and Brownian Motion*. Grundlehren der mathematischen Wissenschaften. 293. Springer-Verlag.
- Santos, Tano and Pietro Veronesi. 2008. Habit Formation, the Cross Section of Stock Returns and the Cash-Flow Risk Puzzle. Working paper, University of Chicago.
- Wachter, Jessica. 2005. Solving Models with External Habit. *Finance Research Letters* 2:210–226.

Wong, Eugene. 1964. The Construction of a Class of Stationary Markoff Processes. In *Stochastic Processes in Mathematical Physics and Engineering*, edited by Richard E. Bellman, no. 17 in Proceedings of Symposia in Applied Mathematics, 264–276. American Mathematical Society.