Topic 1: Markov Chains and Asset Valuation

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Economic problem

- How are asset prices determined in an equilibrium model?
- How is equilibrium valuation linked to absence of arbitrage?
- What are the short- and long-run implications for valuation?
- How can we empirically verify the asset pricing implications of an equilibrium model?

Tools

- Linear algebra and matrix operations, Perron–Frobenius theorem
- Markov chains
- GMM estimation

Literature

Textbook

- Ljungqvist and Sargent (2020), Chapter 2 (Sections 2.2–2.3, Markov chains), Chapters 14 and 15 (asset pricing).

Generalized method of moments

- Hansen (1982), Hansen (2008).

Asset pricing applications

- Lucas (1978), Hansen and Singleton (1982), Hansen and Singleton (1983), Mehra and Prescott (1985).

QuantEcon

- Quantitative Economics with Python: Topic 3 (linear algebra), Topic 25 (finite Markov chains), Topics 74–76 (asset pricing applications in finite state models).
- Advanced Quantitative Economics with Python: Topics 34–35 (more advanced asset pricing applications).

Problem setting

An infinite-horizon endowment economy (Lucas (1978))

- time is discrete and infinite, $t = 0, 1, 2, \ldots$
- every period t, one of a finite number of states $x_t \in \mathcal{X}$ can be realized
- history of states $x^t = (x_0, x_1, \ldots, x_t)$, conditional probability $P\left(x^t | x_0\right)$
- exogenously given aggregate endowment $Y_t = Y\left(x^t\right)$

A representative utility-maximizing investor (von Neumann and Morgenstern (1947), Savage (1954))

- receives individual endowment $y_t = y\left(x^t\right)$
- trades a set of N assets with prices $Q_t^n = Q^n\left(x^t\right)$ and promised cash flows $G_t^n = G^n\left(x^t\right)$, $n = 1, \ldots, n$
- subjective beliefs described by probability $P^i\left(x^t | x_0\right)$, potentially distinct from $P\left(x^t | x_0\right)$
- separable utility over consumption $u\left(c\right)$ with usual properties and time preference β

Investor's problem

For simplicity restrict attention to just one asset.

Investor chooses state-dependent consumption $c_t = c\left(x^t\right)$ and quantity of asset $b_t = b\left(x^t\right)$

$$\max_{\{c_t, b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{x^t} P^i\left(x^t | x_0\right) \beta^t u\left(c_t\right)$$

subject to the sequence of budget constraints, $t = 0, 1, 2, \ldots$

$$c_t(x^t) + b_t(x^t)Q(x^t) = b_{t-1}(x^{t-1})\left[Q(x^t) + G(x^t)\right] + y(x^t)$$

and no-Ponzi conditions that prevent asymptotic overaccumulation of financial liabilities.

Measurability restrictions

- choices made at time t and constraints imposed at time t can only depend on information x^t observed up to time t

Observations

- Problem depends on subjective belief $P^i\left(x^t | x^0\right)$, not on the data-generating probability $P\left(x^t | x_0\right)$

Impose Lagrange multiplier $P^i\left(x^t|x_0\right)\beta^t\mu_t\left(x^t\right)$ on the time t constraint

$$\begin{aligned}\mathcal{L}\left(x_0\right) &= \sum_{t=0}^{\infty}\sum_{x^t} P^i\left(x^t|x_0\right)\beta^t u\left(c_t\right) + \\ &+\sum_{t=0}^{\infty}\sum_{x^t} P^i\left(x^t|x_0\right)\beta^t\mu_t(x^t)\left[b_{t-1}(x^{t-1})\left[Q(x^t)+G(x^t)\right]+y(x^t)-c_t(x^t)-b_t(x^t)Q(x^t)\right]\end{aligned}$$

Optimality conditions

$$\begin{aligned}\left[c_t(x^t)\right] &: \quad u'\left(c_t(x^t)\right)=\mu_t(x^t) \\ \left[b_t(x^t)\right] &: \quad P^i\left(x^t|x_0\right)\beta^t\mu_t(x^t)Q(x^t)=\sum_{x^{t+1}|x^t} P^i\left(x^{t+1}|x_0\right)\beta^{t+1}\mu_{t+1}(x^{t+1})\left[Q(x^{t+1})+G(x^{t+1})\right]\end{aligned}$$

plus transversality conditions.

Combining the optimality conditions and noticing that

$$P^i\left(x^{t+1}|x^t\right) = \frac{P^i\left(x^{t+1}|x_0\right)}{P^i\left(x^t|x_0\right)}$$

we obtain the Euler equation

$$Q(x^t) = \sum_{x^{t+1}|x^t} P^i\left(x^{t+1}|x^t\right)\beta\frac{u'\left(c_{t+1}\left(x^{t+1}\right)\right)}{u'\left(c_t\left(x^t\right)\right)}\left[Q(x^{t+1}) + G(x^{t+1})\right]$$

Dropping arguments, the Euler equation can be written as

$$Q_t = E_t^i\left[\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left(Q_{t+1} + G_{t+1}\right)\right] \tag{1.1}$$

where $E_t^i[\cdot]$ is the conditional expectations operator under probability P^i conditional on time-t information.

Euler equation (one for each traded asset $n = 1, \ldots, N$)

$$Q_t^n = E_t^i \Bigg[\underbrace{\beta \frac{u'(c_{t+1})}{u'(c_t)}}_{s_{t+1}} \left(Q_{t+1}^n + G_{t+1}^n \right) \Bigg]$$

The process S_t with increments

$$s_{t+1} = \frac{S_{t+1}}{S_t} = \beta \frac{u'\left(c_{t+1}\left(x^{t+1}\right)\right)}{u'\left(c_t\left(x^t\right)\right)}$$

that represent the marginal rate of substitution between states x^t and x^{t+1} is called the stochastic discount factor (SDF).

- existence of such a strictly positive SDF is a general property of no-arbitrage markets
- existence of SDF does not require utility-maximizing investors but utility maximization associates the SDF with investors' marginal rate of substitution

Assume traded assets are in zero net supply (this is without loss of generality).

A competitive equilibrium in this endowment economy consists of the endowment process $Y(x^t)$, cash flows $G^n(x^t)$, the price processes $Q^n(x^t)$, and allocations $c_t(x^t)$, $b_t^n(x^t)$ such that

1. Given prices $Q^n(x^t)$, cash flows $G^n(x^t)$, $n = 1, \ldots, n$, and individual endowment $y(x^t)$, the investor chooses consumption $c_t(x^t)$ and portfolio allocation $b_t^n(x^t)$, $n = 1, \ldots, n$, that solve the utility maximization problem;
2. The individual investor is representative

$$c_t(x^t) = C_t(x^t) \qquad y\left(x^t\right) = Y(x^t),$$

3. Markets clear:

$$\begin{aligned} c_t(x^t) &= y(x^t) = C_t(x^t) = Y(x^t) \\ b_t^n(x^t) &= 0 \qquad n = 1, \ldots, N. \end{aligned}$$

Individual and aggregate variables (consumption and endowment) coincide because of the representative agent assumption (the “little k, big K” analogy, Ljungqvist and Sargent (2020), Chapters 8 and 13).

In the definition of the equilibrium

- the dynamics of the state x_t and hence aggregate endowment depend on the data-generating probability measure $P\left(x^t|x_0\right)$
- given state x_t the asset price $Q\left(x_t\right)$ is computed using recursion (1.1) which depends on the subjective belief $P^i\left(x^t|x_0\right)$
- this is a critical distinction

We now impose a rational expectations assumption (Muth (1961), Lucas (1972))

- subjective belief $P^i\left(x^t|x_0\right)$ is identical to the data-generating measure $P\left(x^t|x_0\right)$
- we will return to this implications of this assumption later

Stocks

- cash flow G_t is the dividend flow, ex-dividend price Q_t

$$\text{return } R_{t+1} = \frac{Q_{t+1} + G_{t+1}}{Q_t} \qquad \text{expected return } E_t\left[R_{t+1}\right]$$

Bonds

- cash flow G_t are coupons in $t+1, \ldots, t+T$, and principal at time $t+T$, bond price $Q_t^{[T]}$
- T-period zero-coupon bond: only principal $G_{t+T} = 1$ at time $t+T$

$$\text{yield to maturity } y_t^{[T]} = -\frac{1}{T} \log Q_t^{[T]}$$

- one-period risk-free bond: equal to $G_{t+1} = 1$, zero otherwise, implying $Q_{t+1} = 0$

$$\text{risk-free return (rate) } R_{t+1}^f = \frac{Q_{t+1} + G_{t+1}}{Q_t} = \frac{1}{Q_t} = \left(E_t\left[\frac{S_{t+1}}{S_t}\right]\right)^{-1}$$

Asset values are derived recursively:

$$Q_t = E_t\left[\frac{S_{t+1}}{S_t}G_{t+1}\right] + E_t\left[\frac{S_{t+1}}{S_t}Q_{t+1}\right] = E_t\left[\frac{S_{t+1}}{S_t}G_{t+1} + \frac{S_{t+2}}{S_t}G_{t+1}\right] + E_t\left[\frac{S_{t+2}}{S_t}Q_{t+2}\right].$$

Iterating forward

$$Q_t = \underbrace{\lim_{T\to\infty}\sum_{j=1}^{T} E_t\left[\frac{S_{t+j}}{S_t}G_{t+j}\right]}_{\text{fundamental value}} + \underbrace{\lim_{T\to\infty} E_t\left[\frac{S_{t+T}}{S_t}Q_{t+T}\right]}_{\text{rational bubble}}.$$

Rational bubbles can emerge in specific models (Bewley (1980), Tirole (1985)) but rational expectations equilibria put strong discipline on when this can happen (Santos and Woodford (1997)).

- money is one example in which an asset is valuable for other purposes than delivering cash flows (liquidity purposes, insurance of idiosyncratic risk, intergenerational insurance in OLG models)

Empirical implications

In order to assess the model implications, we need to specify exogenous components

- endowment process, equal, in equilibrium, to the consumption of representative agent: $c_t = Y_t$
- cash flows G_t of traded assets
- preferences of the representative agent: β, $u(\cdot)$

The model makes predictions for the dynamics of asset prices Q_t that can be compared to data.

- in order to fully solve the model, we need to put tractable structure on the dynamics of the underlying state x_t

Assume representative investor with constant relative risk aversion (CRRA) preferences

$$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, \qquad 1 \neq \gamma > 0$$
$$u(c) = \log(c), \qquad \gamma = 1$$

Then the SDF is given by

$$\frac{S_{t+1}}{S_t} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}.$$

- parameters β and γ are to be determined
- consumption growth c_{t+1}/c_t can be measured directly in the data
- under the representative agent assumption, $c_{t+1}/c_t = C_{t+1}/C_t$.

Testable implications of Euler equations

Divide Euler equation (1.1) by Q_t^n to obtain

$$1 = E_t\left[\frac{S_{t+1}}{S_t}\underbrace{\frac{Q_{t+1}^n + G_{t+1}^n}{Q_t^n}}_{R_{t+1}^n}\right]$$

- R_{t+1}^n is the one-period return on asset n

With the CRRA model of preferences, we obtain a set of restrictions

$$1 = E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{t+1}^n\right] \qquad n = 1, \ldots, N \tag{1.2}$$

- collect data on one-period returns R_{t+1}^n and aggregate consumption growth C_{t+1}/C_t
- attempt to find parameters β and γ so that equations (1.2) are satisfied

Recall the expression for (conditional) covariance of two random variables:

$$Cov_t\left(X_{t+1}, Y_{t+1}\right) = E_t\left[X_{t+1}Y_{t+1}\right] - E_t\left[X_{t+1}\right]E_t\left[Y_{t+1}\right]$$

Take a risky return R^n_{t+1} (e.g., stock return), and the return on a one-period risk-free bond (risk-free rate), denoted R^f_{t+1}, subtract the two Euler equations, and apply the above formula:

$$0 = E_t\left[\frac{S_{t+1}}{S_t}\left(R^n_{t+1} - R^f_{t+1}\right)\right] = E_t\left[\frac{S_{t+1}}{S_t}\right]E_t\left[R^n_{t+1} - R^f_{t+1}\right] + Cov_t\left(\frac{S_{t+1}}{S_t}, R^n_{t+1} - R^f_{t+1}\right)$$

Reorganizing,

$$E_t\left[R^n_{t+1} - R^f_{t+1}\right] = -\left(E_t\left[\frac{S_{t+1}}{S_t}\right]\right)^{-1} Cov_t\left(\frac{S_{t+1}}{S_t}, R^n_{t+1} - R^f_{t+1}\right).$$

Interpretation of Euler equations

Euler equations yield a relationship between expected excess returns and covariances with SDF

$$E_t\left[R^n_{t+1} - R^f_{t+1}\right] = -\underbrace{R^f_{t+1}}_{\approx 1} Cov_t\left(\frac{S_{t+1}}{S_t}, R^n_{t+1} - R^f_{t+1}\right).$$

Recall the CRRA SDF

$$\frac{S_{t+1}}{S_t} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

- ‘bad’ states at $t+1$ with low consumption C_{t+1} imply high realizations of the SDF (high MRS)
- vice versa for ‘good’ states
- an asset that delivers low excess returns $R^n_{t+1} - R^f_{t+1}$ in ‘bad’ states and high returns in ‘good’ states will have a negative covariance with the SDF
- such assets are risky: deliver low returns when consumption is valuable
- investors must be compensated for this risk with higher expected returns

Euler equations (1.2) have to hold conditionally (with $E_t[\cdot]$) but they can be conditioned down:

$$0 = E\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R^n_{t+1} - 1\right] \qquad n = 1, \ldots, N.$$

Replace unconditional mean with sample average

- follows from the law of large numbers
- valid when we assume that consumption growth and returns follow stationary processes that satisfy integrability restrictions (finite moment restrictions)

$$0 = \frac{1}{T}\sum_{t=0}^{T-1}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R^n_{t+1} - 1\right] \qquad n = 1, \ldots, N.$$

Using conditioning information

Additional restrictions can be obtained using conditioning information.

$$0 = E_t \left[\frac{S_{t+1}}{S_t} R^n_{t+1} - 1 \right]$$

- conditions from previous slide test whether this relationship holds unconditionally ($E[\cdot]$)
- but what if we are concerned that the conditional equation does not hold?
- then we should be able to predict when it does not hold: denote the predictor variable z_t

The variable z_t (must be in the time-t information set) is called an instrument.

- multiply Euler equation with z_t and apply unconditional expectation

$$0 = E\Bigg[\underbrace{z_t}_{\text{instrument}} \underbrace{\left(\frac{S_{t+1}}{S_t} R^n_{t+1} - 1 \right)}_{\text{Euler equation error}} \Bigg]$$

- these unconditional expectations can again be implemented using time-series averages
- z_t can for example by a cyclical variable to test systematic violations over the business cycle

Implementation using Generalized Method of Moments

Generalized Method of Moments (Hansen (1982)) provides a formal econometric test

- how to find parameters β and γ that yield the best fit to the instrumented Euler equations?
- how to conduct inference using sample data about the validity of the theoretical restrictions?

Imagine instruments z_t^k, $k = 1, \ldots K$, and returns R_{t+1}^n, $n = 1, \ldots, N \implies$ total M moments

- denote the vector of (unknown) parameters $\theta = (\beta, \gamma)$, data $x_{t+1} = (z_t, C_{t+1}/C_t, R_{t+1})$, and

$$f_m(x_{t+1}; \theta) = z_t^k \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1}^n - 1 \right), \qquad m = 1, \ldots, M$$

Denoting $f(x_{t+1}; \theta)$ the column vector with elements $f_m(x_{t+1}; \theta)$ yields the vector moment condition

$$0 = E\left[f(x_{t+1}; \theta)\right].$$

- asset pricing implementation in Hansen and Singleton (1982).

Denote θ_0 the true parameter value, in the sense that it is the unique solution to $0 = E\left[f\left(x_{t+1};\theta\right)\right]$.

The solution can be equivalently found by solving

$$\theta_0 = \min_{\theta} E\left[f\left(x_{t+1};\theta\right)\right]' WE\left[f\left(x_{t+1};\theta\right)\right] \tag{1.3}$$

- W is a positive definite weighting matrix, which implies that the right-hand side is strictly positive whenever $E\left[f\left(x_{t+1};\theta\right)\right]$ is nonzero

In a finite data sample, we replace the theoretical $E\left[f\left(x_{t+1};\theta\right)\right]$ with sample average

$$g_T\left(\theta\right) = \frac{1}{T}\sum_{t=0}^{T-1} f\left(x_{t+1};\theta\right)$$

and solve

$$\hat{\theta}_T = \min_{\theta} g_T\left(\theta\right)' Wg_T\left(\theta\right). \tag{1.4}$$

Consider the covariance matrix

$$V = \sum_{j=-\infty}^{\infty} E\left[f\left(x_{t+1};\theta_0\right) f\left(x_{t+1+j};\theta_0\right)'\right]$$

- this is the long-run covariance matrix of the moment conditions

Hansen (1982) shows that in this case

$$T g_T\left(\theta_0\right)' V^{-1} g_T\left(\theta_0\right) \rightarrow \chi^2\left(M\right) \tag{1.5}$$

- moment conditions that have a lot of variability are downweighed in the objective function

Choice of the weighting matrix W can in practice be a complex issue.

1. Theoretical V involves an infinite-horizon covariance structure which must be approximated (Newey and West (1987)).
   Fortunately, our time-series setup that uses conditional moments simplifies V:

$$\text{for } j \geq 1 : E\left[f(x_{t+1};\theta_0)\, f(x_{t+1+j};\theta_0)'\right] = E\Big[f(x_{t+1};\theta_0)\, \underbrace{E_{t+j}\left[f(x_{t+1+j};\theta_0)'\right]}_{=0}\Big] = 0$$

   so the covariance matrix simplifies to

$$V = E\left[f(x_{t+1};\theta_0)\, f(x_{t+1};\theta_0)'\right] \approx \frac{1}{T}\sum_{t=0}^{T-1} f(x_{t+1};\theta_0)\, f(x_{t+1};\theta_0)'$$

2. The theoretical V is a function of the true parameter θ_0, which is a priori unknown. Hansen et al. (1996) propose a continuously updated estimator

$$\hat{\theta}_T = \min_{\theta} g_T(\theta)' V_T(\theta)^{-1} g_T(\theta)$$

where

$$V_T(\theta) = \frac{1}{T} \sum_{t=0}^{T-1} f(x_{t+1}; \theta) f(x_{t+1}; \theta)' \tag{1.6}$$

3. A two-step procedure is asymptotically valid as well because V can be replaced by its consistent estimator. For example:

- compute $\hat{\theta}_T$ by minimizing (1.4) for some positive definite W
- compute $V\left(\hat{\theta}_T\right)$ using (1.6)
- evaluate the left-hand-side of (1.5) using $\theta_0 = \hat{\theta}_T$

In principle, we could also use the estimated $V\left(\hat{\theta}_T\right)$ as a new W in the minimization (1.4) and obtain a new estimate $\hat{\theta}_T$.

4. In theory, using $V = V(\theta_0)$ as the weighting matrix is asymptotically efficient. But there are obstacles:
    - θ_0 must be estimated using a finite sample of data, so we can at best use $V\left(\hat{\theta}_T\right)$ instead
    - in practical applications, $V\left(\hat{\theta}_T\right)$ can be hard to estimate, which may lead to fragility that puts excessive emphasis on a small number of moments
    - this issue is further magnified when misspecifications are present
    - asset pricing applications are prone to such fragility
    - various aproaches exist how to modify the weighting matrix to deal with these issues

For more discussion on GMM in the time-series context see, for example, Hansen (2001, 2008).

GMM estimation required very little structure, beyond regularity conditions on the estimator.

- we took consumption and returns as data, and estimated preference parameters
- much of the structure of the model was put aside (probability distributions)
- this approach is called partial identification

The GMM estimation approach also critically relied on rational expectations

- the original Euler equation (1.1) depended on the subjective expectation $E_t^i[\cdot]$
- GMM replaced the expectations operators with time-series averages
- but time-series averages approximate expectations operators under the data-generating measure $E_t[\cdot]$
- rational expectations impose $E_t[\cdot] = E_t^i[\cdot]$; without this assumption, we would need to impose other conditions how to link $E_t^i[\cdot]$ and $E_t[\cdot]$

Numerical implementation

We now want to use the model to explicitly solve for asset prices $Q(x_t)$ and their dynamics.

In order to do that, we need to put explicit structure on

- the probabilities $P(x^t|x_0)$ and $P^i(x^t|x_0)$
- mapping from x^t to $C(x^t)$

A Markov chain structure is a computationally tractable approach

- the state x_t follows a Markov process: $P(x_{t+1}|x^t) = P(x_{t+1}|x_t)$
- growth rates of consumption and cash flows are functions of the Markov state

Markov chains and transition probabilities

A time-invariant finite-state Markov chain x_t is characterized by

- a set of states $\mathcal{X} = \{e_1, \ldots, e_n\}$ where e_i are coordinate vectors
- a transition matrix $\mathbf{P}$ with elements

$$\mathbf{P}_{ij} = P\left(x_{t+1} = e_j | x_t = e_i\right)$$

- a time-0 distribution $\pi_0 = P\left(x_0\right)$
- investors endowmed with correct beliefs $\mathbf{P}^I = \mathbf{P}$, $\pi_0^I = \pi_0$.

While this is not necessary, it is convenient to assume that $\mathbf{P}$ has strictly positive elements.

- then x_t has a unique stationary distribution to which x_t converges

For more details, see

- Ljungqvist and Sargent (2020), Chapter 2 (Section 2.2)
- QuantEcon https://python.quantecon.org/finite_markov.html

Conditional probability $\pi_{t,j} = P(x_t = e_j)$ can be computed as

$$\pi_{t,j} = P(x_t = e_j) = \sum_{i=1}^{n} P(x_t = e_j | x_{t-1} = e_i) P(x_{t-1} = e_i) = \sum_{i=1}^{n} \mathbf{P}_{ij} \pi_{t-1,i} = \pi'_{t-1} \mathbf{P}_{\cdot j}$$

- in stacked form

$$\pi'_t = \pi'_{t-1} \mathbf{P}$$

- multiperiod conditional probabilities obtained iteratively

$$\pi'_t = \pi'_{t-1} \mathbf{P} = \pi'_{t-2} \mathbf{P}^2 = \ldots = \pi'_0 \mathbf{P}^t$$

  which also implies that $\left[\mathbf{P}^t\right]_{ij} = (x_t = e_j | x_0 = e_i)$.
- when $\mathbf{P}$ has strictly positive elements, then

$$\lim_{t \to \infty} \pi'_t = \lim_{t \to \infty} \pi'_0 \mathbf{P}^t = \pi'$$

  where π is the unique stationary density, $\pi' = \pi' \mathbf{P}$.

Let $y_t = y' x_t$ be a random variable, a function of the Markov chain.

- conditional expectations of x_{t+1} are given by

$$E\left[x_{t+1}|x_t = e_i\right] = \sum_{j=1}^{n} e_j P\left(x_{t+1} = e_j|x_t = e_i\right) = \sum_{j=1}^{n} e_j \mathbf{P}_{ij} = \mathbf{P}' e_i$$

- conditional expectations of y_{t+1}

$$E\left[y_{t+1}|x_t = e_i\right] = y' E\left[x_{t+1}|x_t = e_i\right] = y' \mathbf{P}' e_i = e_i' \mathbf{P} y$$

- longer-horizon conditional expectations can be computed iteratively

$$E\left[y_{t+j}|x_t = e_i\right] = e_i' \mathbf{P}^j y$$

- asymptotically, we obtain the unconditional expectation

$$\lim_{j\to\infty} E\left[y_{t+j}|x_t = e_i\right] = \lim_{j\to\infty} e_i' \mathbf{P}^j y = \pi' y$$

Assume that the stochastic discount factor can be written as

$$\log S_{t+1} - \log S_t = g_S\left(x_t, x_{t+1}\right)$$

- incorporates a large class of models of interest
- denote Γ_S the $n \times n$ matrix with elements

$$\left[\Gamma_S\right]_{ij} = \exp\left(g_S\left(x_t = e_i, x_{t+1} = e_j\right)\right)$$

CRRA stochastic discount factor

- stationary consumption $C_t = C\left(x_t\right)$

$$\left[\Gamma_S\right]_{ij} = \beta \left(\frac{C\left(x_{t+1} = e_j\right)}{C\left(x_t = e_i\right)}\right)^{-\gamma}$$

- stationary consumption growth

$$\begin{aligned} \log C_{t+1} - \log C_t &= g_C\left(x_t, x_{t+1}\right) \\ \log S_{t+1} - \log S_t &= \log \beta - \gamma g_C\left(x_t, x_{t+1}\right) \end{aligned}$$

Denote $Q_t^{[T]}$ the time-t price of an asset that pays $G_{t+T} = G(x_{t+T})$ at time $t+T$.

- an example of a stationary payoff, special case $G_{t+T} = 1$ a zero-coupon bond
- asset prices will have Markov structure, $Q_t^{[T]} = Q^{[T]}(x_t)$, and can be computed iteratively:

$$Q^{[1]}(x_t) = E_t^i\left[\exp\left(g_S(x_t, x_{t+1})\right) G(x_{t+1})\right]$$

and, for $T = 1, 2, \ldots$

$$Q^{[T+1]}(x_t) = E_t^i\left[\exp\left(g_S(x_t, x_{t+1})\right) Q^{[T]}(x_{t+1})\right]$$

Denote the $\mathbf{q}^{[T]}$ the vector of state-dependent prices

$$\left[\mathbf{q}^{[T]}\right]_i = Q^{[T]}(x_t = e_i)$$

Using the matrix structure, the recursive equation can be expressed as

$$\left[\mathbf{q}^{[T+1]}\right]_i = Q^{[T+1]}(x_t = e_i) = \sum_{j=1}^{n} [\mathbf{P}]_{ij} [\Gamma_S]_{ij} \left[\mathbf{q}^{[T]}\right]_j$$

This can be written in compact form as

$$\mathbf{q}^{[T+1]} = (\mathbf{P} * \Gamma_S)\, \mathbf{q}^{[T]}$$

where $\mathbf{P} * \Gamma_S$ is the element-wise multiplication of the two matrices.

In many models, cash flows can be non-stationary, with stationary growth rates.

To incorporate this, rewrite the Euler equations (1.1) as

$$\frac{Q_t}{G_t} = E_t^i \left[\frac{S_{t+1}}{S_t} \left(\frac{Q_{t+1}}{G_{t+1}} + 1 \right) \frac{G_{t+1}}{G_t} \right] \tag{1.7}$$

Let us assume that we can write

$$\begin{aligned} \log G_{t+1} - \log G_t &= g_G\left(x_t, x_{t+1}\right) \\ \left[\Gamma_G\right]_{ij} &= \exp\left(g_G\left(x_t = e_i, x_{t+1} = e_j\right)\right) \end{aligned}$$

Then we can conclude that the price-dividend ratio is stationary:

$$\frac{Q_t}{G_t} = q\left(x_t\right)$$

- with this guess, the right-hand side in (1.7) is an expectation over x_{t+1} with conditional probability that depends on x_t
- this validates the conclusion that the left-hand side is only a function of x_t

The matrix implementation is analogous to the case with stationary cash flows.

Denote the $\mathbf{q}$ the vector of state-dependent price-dividend ratios

$$[\mathbf{q}]_i = q\left(x_t = e_i\right).$$

Using the matrix structure, the recursive equation (1.7) can be expressed as

$$[\mathbf{q}]_i = q\left(x_t = e_i\right) = \sum_{j=1}^{n} [\mathbf{P}]_{ij} \left[\Gamma_S\right]_{ij} \left[\Gamma_G\right]_{ij} \left([\mathbf{q}]_j + 1\right)$$

This can be written in compact form as

$$\mathbf{q} = \underbrace{\left(\mathbf{P} * \Gamma_S * \Gamma_G\right)}_{=\mathbf{M}} (\mathbf{q} + \mathbf{1}). \tag{1.8}$$

The solution for this equation is the fixed point (assuming $\mathbf{M}$ is a stable matrix):

$$\mathbf{q} = (\mathbf{I} - \mathbf{M})^{-1} \mathbf{M}\mathbf{1}$$

- can also be obtained by iterations on (1.8) start with any initial guess $\mathbf{q}^0$

Example: Equity premium puzzle

Mehra and Prescott (1985) construct an endowment economy with a 2-state Markov chain for the growth rate of aggregate endowment.

- assume rational expectations, $\mathbf{P} = \mathbf{P}^i$, with

$$\mathbf{P} = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix}$$

- assume CRRA utility function and vary the risk aversion parameter γ
- the growth rate of aggregate endowment is given by

$$\Gamma_C = \begin{bmatrix} 1+\mu+\delta & 1+\mu-\delta \\ 1+\mu+\delta & 1+\mu-\delta \end{bmatrix}$$

  state e_1 is the high-growth state, state e_2 is the low-growth state
- calibrate annual parameters $\mu = 0.018$, $\delta = 0.036$, $\phi = 0.43$

Compute the price-dividend ratio on the claim on aggregate endowment (wealth-consumption ratio) $Q_t/C_t = q(x_t)$ and compute the 'equity' return

$$R^c_{t+1} = \frac{Q_{t+1} + C_{t+1}}{Q_t} = \frac{q(x_{t+1}) + 1}{q(x_t)} \frac{C_{t+1}}{C_t} \qquad \Longrightarrow \qquad \mathbf{R}^c_{ij} = \frac{\mathbf{q}_j + 1}{\mathbf{q}_i} [\Gamma_C]_{ij}$$

and the risk-free rate

$$R^f_{t+1} = \left(E_t \left[\frac{S_{t+1}}{S_t} \right] \right)^{-1} \qquad \Longrightarrow \qquad \mathbf{R}^f_{ij} = \left(\left[\mathbf{P}^i * \Gamma_S \right]_{i\cdot} \mathbf{1} \right)^{-1}.$$

The conditional equity premium is the difference between expectations of the two returns:

$$E_t \left[R^c_{t+1} - R^f_{t+1} | x_t = e_i \right] = \sum_{j=1}^{2} \left(\mathbf{R}^c_{ij} - \mathbf{R}^f_{ij} \right) \mathbf{P}_{ij}$$

and the unconditional premium, under the unconditional distribution π,

$$E \left[R^c_{t+1} - R^f_{t+1} \right] = \sum_{i=1}^{2} \pi_i \sum_{j=1}^{2} \left(\mathbf{R}^c_{ij} - \mathbf{R}^f_{ij} \right) \mathbf{P}_{ij}.$$

Mehra and Prescott (1985) vary parameters $\beta \in (0, 1)$, $\gamma \in (0, 10)$ and compute resulting combinations of the risk-free rate and the risk premium

- find that the model is vastly inconsistent with data $\implies$ equity premium puzzle
- same conclusion reached in Hansen and Singleton (1983)

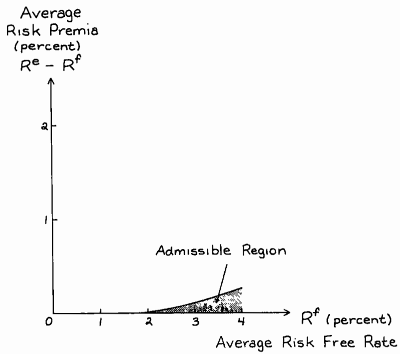

Fig. 4. Set of admissible average equity risk premia and real returns.

Fundamental theorem of asset pricing

In the endowment economy, we derived valuation formulas in the form of expected discounted values of cash flows

- the stochastic discount factor was associated with investor's marginal rate of substitution

However, the existence of some stochastic discount factor is not restricted to markets with utility maximizing agents.

- the central link is absence of arbitrage (Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981))

We restrict attention to a two-period market with K traded securities. There is a single state at time t and n possible states x_{t+1} at time $t+1$.

- probability distribution given by an $1 \times n$ vector p
- time-$t+1$ cash flows $G^k_{t+1} = G^k(x_{t+1})$, $k = 1, \ldots, K$, stacked as rows into a $K \times n$ matrix $\mathbf{G}$
- time-t prices Q^k_t, stacked into a $1 \times K$ vector $\mathbf{Q}$

At time t, agent chooses a portfolio θ of the securities (a $K \times 1$ vector).

- time-$t+1$ payoff $\theta' \mathbf{G}$, time-t price $\mathbf{Q}\theta$.

Definition 1.1

An arbitrage is a portfolio θ such that either

1. $\mathbf{G}\theta \in R_+ \setminus \{0\}$ and $\mathbf{Q}\theta \leq 0$; or
2. $\mathbf{G}\theta \in R_+$ and $\mathbf{Q}\theta < 0$.

In words, an arbitrage is a portfolio that allows to generate a nonnegative payoff with strictly positive payoffs in some states out of a nonpositive investment, or a nonnegative payoff out of a strictly negative investment.

- existence of such arbitrages is inconsistent with investor optimization in the equilibrium model studied earlier

Definition 1.2

A stochastic discount factor $s_{t+1} = s\left(x_{t+1}\right)$, represented by an $n \times 1$ vector $\mathbf{s}$, is a strictly positive random variable such that

$$\mathbf{Q}^k = \sum_{j=1}^{n} \mathbf{p}_j \mathbf{s}_j \mathbf{G}_{kj} \qquad k = 1, \ldots, K$$

Theorem 1.3 (Fundamental theorem of asset pricing)

A stochastic discount factor exists if and only if there is no arbitrage.

Interpretation of the Fundamental theorem of asset pricing

SDF represents marginal valuation of payoffs in alternative states $j = 1, \ldots, N$

- the valuation formulas

$$\mathbf{Q}^k = \sum_{j=1}^{n} \mathbf{p}_j \mathbf{s}_j \mathbf{G}_{kj}$$

  are consistency conditions: prices $\mathbf{Q}$ of more complicated payoffs are represented as linear aggregates of prices of payoffs in individual states
- when there is no arbitrage, prices $\mathbf{Q}$ must be consistent in the sense that they allow such representation
- SDF may not be unique when markets are incomplete, will be unique in complete markets.

Equilibrium in the competitive market guarantees the existence of an SDF

- consequence of the optimal choice of a utility-maximizing investor
- existence of arbitrage would imply that solution to the investor's problem does not exist $\Longrightarrow$ no equilibrium

Summary

The finite-state Markov chain environment served as a laboratory that allowed to

- outline foundations of asset pricing theory
- provide tractable calculations using finite-state Markov chains
- test valuation equations implied by utility-maximizing agents using GMM

Generalizations

- theoretical results carry over to more complicated setups, with continuous state spaces, continuous time, and more sophisticated underlying structure of the economy
- numerical solutions of such economies often rely on discretizations which translate the model to a finite-state approximation
- in such approximations, numerical methods based on Markov chains apply again

Appendix

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