Topic 3: Perturbation methods and linear state space models

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Economic problem

- How do we capture joint dynamics of many state variables in a tractable way?
- What are the tradeoffs between tractability and ability of the model to accurately capture true underlying dynamics?

Tools

- Linear vector autoregressions
- Perturbation approximations leading to linear dynamics

Textbook

- Ljungqvist and Sargent (2020), Chapter 2 (Sections 2.4–2.5): Linear vector autoregressions, Chapter 3: Applications, Chapter 6: Linear quadratic dynamic programming
- Judd (1998), Chapter 13 (perturbation methods)

QuantEcon

- Quantitative Economics with Python: Topic 24 (AR(1) processes), Topic 27–28 (linear state space models), Topic 78 (linear regression)
- Advanced Quantitative Economics with Python: Topics 16–19 (dynamic linear economies)

Problem setting

In multidimensional environments, we face computational tradeoffs.

- global approximations focus on capturing nonlinear dynamics
- multidimensional state spaces may cease to be tractable (curse of dimensionality)

If capturing joint dynamics of many state variables is essential, we may need to compromise.

- restrict attention to (semi)analytical dynamics under tractable functional forms ...
- ... while sacrificing the capacity to capture some of the nonlinearities

Here, we focus on linear dynamics, and approximations that lead to such dynamics.

- a linear vector-autoregression is a tractable and well-understood model
- it is flexible and scalable to many dimensions while preserving tractability
- linear-quadratic models embed many appealing features in economic analysis

Perturbation methods approximate nonlinear models with linear dynamics

- asset pricing: loglinearization of valuation ratios
- macroeconomics: DSGE modeling

The multivariate stochastic linear model is described by the following components:

- state of the system $x_t \in \mathbb{R}^n$,
- initial distribution $\pi_0(x_0) \sim N(\mu_0, \Sigma_0)$,
- transition density $\pi(x' \mid x) \sim N(A_o x, CC')$, where A_o is an $n \times n$ matrix and C is an $n \times p$ matrix.

The model can be equivalently represented using the stochastic linear difference equation

$$x_{t+1} = A_o x_t + C w_{t+1} \qquad w_{t+1} \sim N(0, I_p) \quad \text{iid.} \tag{3.1}$$

where w_{t+1} is an $p \times 1$ vector of iid Gaussian shocks (so-called random innovations).

Such a model is called a vector autoregression (VAR).

When the VAR equation can be inverted to obtain

$$w_{t+1} = C^{-1}\left(x_{t+1} - A_o x_t\right)$$

then information available at time t can be equivalently expressed using partial histories $x^t = (x_0, x_1, \ldots, x_t)$ or using the histories of innovations $(x_0, w_1, \ldots w_t)$.

- we denote this information set (σ-algebra) $\mathcal{F}_t$

The VAR has a Markov structure

- the distribution of $x_{t+j}, j \geq 1$ conditional on x_t is the same as the distribution conditional on $\mathcal{F}_t$

Some of the results that follow will continue to hold under weaker assumptions.

We can relax the Gaussian assumption, and instead assume that w_{t+1} is a random vector satisfying

$$\begin{aligned} E\left[w_{t+1} \mid \mathcal{F}_t\right] &= 0 \\ E\left[w_{t+1} w_{t+1}' \mid \mathcal{F}_t\right] &= I_p, \end{aligned} \tag{3.2}$$

where $\mathcal{F}_t$ is the σ-algebra (information set) generated by $(x_0, w_1, \ldots w_t)$. The sequence of shocks $\{w_{t+1}\}_{t=0}^{\infty}$ satisfying (3.2) is called a martingale difference sequence.

An even weaker assumption further relaxes the conditional moments, and only assumes that the shocks are unconditionally mean zero and uncorrelated

$$\begin{aligned} E\left[w_{t+1}\right] &= 0 \\ E\left[w_t w_{t-j}'\right] &= I_p \cdot \mathbf{1}\left\{j = 0\right\}. \end{aligned} \tag{3.3}$$

A sequence of shocks satisfying the pair of restrictions (3.3) is called white noise.

Probabilistic setup: Ω the sample space of paths, $\mathcal{F}$ a σ-algebra (sets of paths which can be assigned a probability), $\mathcal{F}_t$ a filtration (sequence of information sets expressing information known at time t), P a probability measure over $\mathcal{F}$

Definition 3.1

An n-dimensional process $\{x_t\}_{t=0}^{\infty}$ on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{\infty}, P\right)$ is a **martingale with respect to the filtration** $\{\mathcal{F}_t\}$ **and the probability measure** P if:

1. x_t is $\mathcal{F}_t$-measurable (i.e., x_t only depends on the information available up to time t.),
2. $E\left[|x_t|\right] < \infty$ for all $t \in \mathcal{T}$,
3. $E\left[x_s|\mathcal{F}_t\right] = x_t$ for all $s \geq t$.

Example: Asset pricing Euler equations

$$Q_t = E_t\left[\frac{S_{t+1}}{S_t}Q_{t+1}\right]$$

The discounted price process $x_t = S_tQ_t$ is a martingale.

We will often append an observation equation, or measurement equation, to obtain what is called a state-space representation of the model:

$$\begin{aligned} x_{t+1} &= A_0 x_t + C w_{t+1} \\ y_t &= G x_t + v_t \end{aligned} \tag{3.4}$$

where y_t and v_t are $m \times 1$ vectors.

- the vector y_t represents observations of a potentially 'hidden' state x_t
- v_t is iid measurement noise with a given covariance matrix.

When x_t is not observable, we will attempt to infer an estimate of the state from observations y_t (a filtering problem).

A scalar second-order autoregression

$$z_{t+1} = \alpha + \rho_1 z_t + \rho_2 z_{t-1} + w_{t+1} \tag{3.5}$$

can be written as

$$x_{t+1} = \begin{bmatrix} z_{t+1} \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 & \alpha \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_t \\ z_{t-1} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w_{t+1}$$

with measurement equation

$$z_t = [1 \ 0 \ 0] \, x_t$$

The matrix A_o constructed above is called the companion form.

The autoregressive model (3.5) is an example of how uncorrelated disturbances w_t may generate persistent oscillations in the observed series z_t.

- This effect was independently described by Eugen Slutsky and Udny Yule in Slutsky (1927) (appeared in English as Slutsky (1937)) and Yule (1927), and is known as the **Slutsky–Yule effect**.
- Slutsky (1927) noted, for example, that moving averages constructed from the random numbers drawn in the Russian government lottery resemble the time series of British business cycles.

The idea revolutionized the way how to think about the propagation mechanism generating business cycles.

- Ragnar Frisch constructed a continuous-time model of aggregate dynamics in Frisch (1933) in which he distinguishes between the 'impulse problem' and the 'propagation problem'.
- Oscillations in his model are generated by a time-to-build mechanism where capital goods need time to be completed before they can be used for production, an early precursor to the time-to-build model of Kydland and Prescott (1982).
- As another early example, the equilibrium dynamics for aggregate output in the multiplier-accelerator model of Samuelson (1939) take exactly the form (3.5), where ρ_1 and ρ_2 are model parameters calibrated to mimic the characteristics of business cycle fluctuations, generated by fluctuations in government spending.

See also https://www.minneapolisfed.org/article/2009/the-meaning-of-slutsky

The vector autoregression can incorporate moving-average dynamics by stacking the history of shocks. The ARMA(1,1) model

$$z_{t+1} = \rho z_t + w_{t+1} + \gamma w_t$$

can be written as

$$x_{t+1} = \begin{bmatrix} z_{t+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & \gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ w_t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_{t+1}$$

with measurement equation

$$z_t = [1 \ 0] \, x_t.$$

Other examples of models that can be suitably stacked into the VAR form include an order-k vector autoregression

$$z_{t+1} = \sum_{j=1}^{k} A_j z_{t+1-j} + C_y w_{t+1},$$

or models that include deterministic or stochastic seasonality:

$$\begin{aligned} y_t &= y_{t-4} \\ y_t &= \phi y_{t-4} + w_t. \end{aligned}$$

When innovations w_{t+1} are normally distributed and the unconditional distribution of the initial state x_0 is normal as well, then the linear form of (3.1):

$$x_{t+1} = A_o x_t + C w_{t+1} \qquad w_{t+1} \sim N\left(0, I_p\right) \quad \text{iid.}$$

implies that x_t will be normally distributed as well.

Since normal distributions are completely described by their first two moments, tracing the first two moments over time is sufficient for the description of the joint distribution of the process.

The dynamics of the first two moments is of interest even when innovations are not normal.

This leads us to the definition of covariance stationarity.

Definition 3.2

A stochastic process is said to be **covariance stationary** if

- the mean is independent of time, $E[x_t] = E[x_0] = \bar{\mu}$
- the sequence of autocovariance matrices

$$E\left[(x_t - E[x_t])(x_{t+j} - E[x_{t+j}])'\right]$$

only depends on j, not on t.

- A stationary process is covariance stationary.
- A linear covariance stationary process with normal innovations and normal unconditional distribution of the initial state is also stationary.

Definition 3.3

A real square matrix A_o is said to be **stable** if all its eigenvalues are strictly within the unit circle.

In order for x_{t+1} to have a stationary mean different from zero, it will often be useful to impose a particular structure on

$$x_{t+1} = A_o x_t + C w_{t+1} \tag{3.6}$$

by singling out a constant from the evolution of the state:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & \widetilde{A} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} 0 \\ \widetilde{C} \end{bmatrix} w_{t+1} \qquad x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} \tag{3.7}$$

where $x_{1,t}$ is scalar. The matrix A_o then has one unit root, and the remaining roots are the roots of $\widetilde{A}$ which we assume is stable.

Denote $\mu_t \doteq E\left[x_t\right]$ the unconditional mean of x_t. Then

$$\mu_{t+1} = A_o \mu_t$$

and we can find $\lim_{t\to\infty} \mu_t = \bar{\mu}$ as the unique solution to

$$\bar{\mu} = A_0 \bar{\mu} \qquad \Longrightarrow \qquad \left(I - A_0\right) \bar{\mu} = 0$$

To provide more information let us look at the structured equation (3.7). Denote

$$\left(\begin{array}{c} \bar{\mu}_1 \\ \bar{\mu}_2 \end{array} \right) = \lim_{t\to\infty} E \left[\begin{array}{c} x_{1,t} \\ x_{2,t} \end{array} \right].$$

Obviously, $x_{1,t} = x_{1,0} = \bar{\mu}_1$.

The lower block of (3.7) can be written as

$$x_{2,t+1} = b x_{1,0} + \widetilde{A} x_{2,t} + \widetilde{C} w_{t+1}.$$

Taking unconditional expectations, we have

$$\mu_{2,t+1} = b\bar{\mu}_1 + \widetilde{A}\mu_{2,t}.$$

When $\widetilde{A}$ is stable, $\lim_{t\to\infty} \mu_{2,t} = \mu_{2,\infty} = \bar{\mu}_2$, and we can find the value as

$$\bar{\mu}_2 = \left(I - \widetilde{A}\right)^{-1} b\bar{\mu}_1.$$

In order to derive the evolution of unconditional variance, denote

$$\Sigma_t \doteq E\left[(x_t - \mu_t)(x_t - \mu_t)'\right].$$

The law of motion for Σ_t can be derived from (3.6) by subtracting the unconditional mean $\mu_{t+1} = A_0 \mu_t$ from both sides and taking the variance of both sides. Hence

$$\Sigma_{t+1} = A_0 \Sigma_t A_0' + CC'.$$

A fixed point of this recursion satisfies

$$\Sigma_\infty = A_0 \Sigma_\infty A_0' + CC'. \tag{3.8}$$

We will denote this fixed point $C_x(0) = \Sigma_\infty$. This fixed point is the covariance matrix

$$C_x(0) = E\left[(x_t - \bar{\mu})(x_t - \bar{\mu})'\right]$$

under the stationary distribution. Equation (3.8) is a discrete Lyapunov equation and can be efficiently solved using alternative algorithms (like the doubling algorithm).

Similarly, to compute the autocovariance function $C_x(j)$, start with (3.6) and write

$$\begin{aligned} x_{t+j} - \mu_{t+j} &= A_o \left(x_{t+j-1} - \mu_{t+j-1} \right) + C w_{t+j} = \ldots \\ &= A_o^j \left(x_t - \mu_t \right) + A_o^{j-1} C w_{t+1} + \ldots + C w_{t+j} \end{aligned}$$

Post-multiply by $(x_t - \mu_t)'$ and take unconditional expectations to obtain

$$E\left[\left(x_{t+j} - \mu_{t+j} \right) \left(x_t - \mu_t \right)' \right] = A_o^j E\left[\left(x_t - \mu_t \right) \left(x_t - \mu_t \right)' \right]$$

Hence, when the process has a stationary mean, we obtain

$$C_x(j) = A_o^j C_x(0) .$$

The sequence $\{C_x(j)\}_{j=0}^{\infty}$ is the autocovariance function or autocovariogram.

To summarize, we distinguish different moments based on the conditioning we impose:

- conditional moments $E\left[x_{t+1} \mid x_t\right] = Ax_t$, $Cov\left(x_{t+1} \mid x_t\right) = CC'$
- moments conditional on x_0,

$$\begin{aligned} E\left[x_t \mid x_0\right] &= E_0\left[x_t\right] = A_o^t x_0 \\ E\left[\left(x_t - E_0\left[x_t\right]\right)\left(x_t - E_0\left[x_t\right]\right)'\right] &= \sum_{h=0}^{t-1} A_o^h CC' \left(A_o^h\right)' \end{aligned}$$

- unconditional moments $E\left[x_t\right] \doteq \mu_t$ and $E\left[\left(x_t - \mu_t\right)\left(x_t - \mu_t\right)'\right] = \Sigma_t$, satisfying

$$\begin{aligned} \mu_{t+1} &= A_o \mu_t \\ \Sigma_{t+1} &= A_o \Sigma_t A_o' + CC' \end{aligned}$$

- stationary moments

$$\begin{aligned} \left(I - A_o\right)\bar{\mu} &= 0 \\ C_x\left(0\right) &= A_o C_x\left(0\right) A_o' + CC' \\ C_x\left(j\right) &= A_o^j C_x\left(0\right) \end{aligned}$$

The stochastic process we posited in (3.1):

$$x_{t+1} = A_o x_t + C w_{t+1} \qquad w_{t+1} \sim N(0, I_p) \quad \text{iid.}$$

specifies a law of motion that describes a deterministic propagation mechanism for x_t, systematically perturbed by random innovations w_{t+1}.

This idea goes back to the impulse and propagation problems described by Frisch (1933):

> *"There are several alternative ways in which one may approach the impulse problem... One way which I believe is particularly fruitful and promising is to study what would become of the solution of a determinate dynamic system if it were exposed to a stream of erratic shocks that constantly upsets the continuous evolution, and by so doing introduces into the system the energy necessary to maintain the swings."*

In order to understand the propagation mechanism, we want to capture how a shock today affects the distribution of the stochastic process in the future.

What are the consequences of perturbing the shock w_1 today for the distribution of x_t, $t \geq 1$?

Consider a common initial condition x_0 and two alternative processes representing iid disturbances:

$$\begin{aligned} w &= \{w_1, w_2, w_3, \ldots\} \\ \widetilde{w} &= \{\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3, \ldots\} \end{aligned}$$

where we impose that $\widetilde{w}_j = w_j$, $\forall j \geq 2$.

The innovation processes thus have the same innovations except period one. Now define

$$\begin{aligned} x_{t+1} &= A_o x_t + C w_{t+1}, \qquad t \geq 0 \\ \widetilde{x}_{t+1} &= A_o \widetilde{x}_t + C \widetilde{w}_{t+1} \end{aligned}$$

The impulse response function is defined as the difference

$$\widetilde{x}_{t+1} - x_{t+1}.$$

Observe that the impulse response function is a stochastic process that in general depends on x_0, as well as on both sequences of shocks $\widetilde{w}$ and w.

In the general nonlinear case, we need to think about ways how to summarize information contained in the process $\widetilde{x}_{t+1} - x_{t+1}$.

It turns out that in the linear case, this difference takes a convenient simple form.

In order to derive it, notice that

$$\begin{aligned} x_{t+1} &= A_o x_t + C w_{t+1} = A_o^2 x_{t-1} + A_o C w_t + C w_{t+1} \\ &= A_o^{t+1} x_0 + \sum_{j=0}^{t} A_o^j C w_{t+1-j} \end{aligned} \tag{3.9}$$

This is the so-called moving-average representation of the process $\{x_t\}$ that specifies the process as a linear combination of past innovations.

Using this moving-average representation, it is easy to infer that

$$\widetilde{x}_{t+1} - x_{t+1} = A_0^t C \left(\widetilde{w}_1 - w_1 \right)$$

$$\widetilde{x}_{t+1} - x_{t+1} = A_0^t C \left(\widetilde{w}_1 - w_1 \right)$$

In the study of linear models, it is a common choice to take $w_1 = 0$ and $\widetilde{w}_1 = e_k$, $k = 1, \ldots, p$. The matrix-valued function

$$h_t = A_o^t C$$

is therefore also commonly referred to as (linear) impulse response function.

- h_t is a matrix whose entry $\left[A_o^t C\right]_{ik}$ represents the response t periods after the impact of the shock of the i-the element of the vector state process x to a perturbation of the k-th element of the innovation w_1

The linear-quadratic formulas used in the computation of conditional expectations and covariances have wide-ranging applications.

- prediction formulas
- present discounted values
- geometric sums of quadratic forms
- asset valuation
- evaluation of dynamic criteria in linear-quadratic models
- optimal control in linear-quadratic models

See Ljungqvist and Sargent (2020) for extensive details

- Chapter 2 (Section 2.4): VAR time-series model
- Chapter 3: wide range of theoretical applications
- Chapter 6: implementation in dynamic programming

The moving average representation is an extremely powerful tool for computing expectations and other statistics. Rewrite (3.9) as

$$x_{t+j} = A_o^j x_t + \sum_{k=0}^{j-1} A_o^k C w_{t+j-k}$$

Hence

$$E_t\left[x_{t+j}\right] \doteq E\left[x_{t+j} \mid x_t\right] = A_o^j x_t.$$

Similarly, consider a function $y_t = G x_t$ where G is a conformable matrix. Then

$$E_t\left[\sum_{j=0}^{\infty} \beta^j y_{t+j}\right] = G \sum_{j=0}^{\infty} \left(\beta A_o\right)^j x_t = G\left(I - \beta A_o\right)^{-1} x_t$$

provided that the matrix βA_o has all unit roots smaller than one in modulus.

In linear-quadratic models, we often want to calculate

$$\alpha_t = E_t \left[\sum_{j=0}^{\infty} \beta^j x'_{t+j} Y x_{t+j} \right]. \tag{3.10}$$

We can proceed by guess and verify and establish a recursive formula

$$\begin{aligned} \alpha_t &= x'_t Y x_t + \beta E_t \left[E_{t+1} \sum_{j=0}^{\infty} \beta^j x'_{t+1+j} Y x_{t+1+j} \right] = \\ &= x'_t Y x_t + \beta E_t \alpha_{t+1}. \end{aligned}$$

Guessing the solution of the form

$$\alpha_t = x_t' \nu x_t + \sigma$$

where ν is an unknown symmetric $n \times n$ matrix and σ an unknown scalar, we plug in to obtain

$$\begin{aligned} x_t' \nu x_t + \sigma &= x_t' Y x_t + \beta E_t \left[x_{t+1}' \nu x_{t+1} + \sigma \right] \\ &= x_t' Y x_t + \beta x_t' A_o' \nu A_o x_t + \beta E_t \left[w_{t+1}' C' \nu C w_{t+1} \right] + \beta \sigma \\ &= x_t' Y x_t + \beta x_t' A_o' \nu A_o x_t + \beta \mathrm{tr} \left[C' \nu C \right] + \beta \sigma . \end{aligned}$$

Comparing coefficients on constant terms and terms involving squares of x_t, we obtain

$$\begin{aligned} \nu &= Y + \beta A_o' \nu A_o \\ \sigma &= (1 - \beta)^{-1} \beta \mathrm{tr} \left[C' \nu C \right] \end{aligned} \qquad (3.11)$$

where $\mathrm{tr}[\cdot]$ denotes the trace of a matrix. The equation for ν is again a discrete Lyapunov equation.

We want to determine the value of an asset as the present discounted value of future cash flows.

- we need to determine a model of cash flows, and a model of discounting

Consider a **cash flow** y_t and a **discount factor** z_t, modeled as

$$y_t = Gx_t \qquad z_t = Hx_t$$

where G and H are row vectors. We are interested in computing the asset price

$$p_t = E_t \left[\sum_{j=0}^{\infty} \beta^j z_{t+j} y_{t+j} \right]$$

- given the **stochastic discount factor** process $\beta^j z_{t+j}$, the asset price is a linear function of the cash flows.
- the stochastic discount factor is typically derived from agent's preferences, reflecting the marginal rate of substitution between today and uncertain future states.

Rewriting the valuation equation as

$$p_t = E_t \left[\sum_{j=0}^{\infty} \beta^j x'_{t+j} H' G x_{t+j} \right],$$

we can use (3.10) to evaluate this sum to obtain

$$p_t = x'_t \nu x_t + \sigma.$$

- the coefficients ν and σ are determined in (3.11).
- σ is determined as the discounted sum of covariances of the innovations in z_t and y_t

$$\begin{aligned} \nu &= H'G + \beta A'_o \nu A_o \\ \sigma &= (1-\beta)^{-1} \beta \mathrm{tr} \left[C' \nu C \right] \end{aligned}$$

- the term $C'\nu C$ depends on the underlying volatility of x_t reflected in C, as well as on $H'G$ that reflects the comovement of cash flows with the stochastic discount factor.
- we can interpret σ as a **risk premium** on asset with cash flow y_t.

Perturbation methods

Linear dynamics are appealing for their tractability.

- allow for easy treatment of high-dimensional problems

Most models do not adhere to such a linear form.

- however, a range of stochastic models can be suitably approximated on an ‘interesting’ part of the state space using linear dynamics

Peturbation method idea

- consider the stochastic model without uncertainty and assume that such a deterministic version of the model converges to a steady state.
- introduce a ‘small’ amount of uncertainty (perturb the deterministic model) and study the approximate ‘local’ behavior of the model in the vicinity of the steady state
- extend the approximate local behavior to the original amount of uncertainty

The method follows from an application of the idea of Taylor's theorem to a dynamic environment.

- consider a function $f(x) : \mathbb{R} \to \mathbb{R}$ and a particular point $\bar{x} \in \mathbb{R}$
- if $f(x)$ is k-times differentiable at $\bar{x}$, then $f(x)$ can be written as

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{f''(\bar{x})}{2}(x - \bar{x})^2 + \ldots + \frac{f^{(k)}(\bar{x})}{k!}(x - \bar{x})^k + h_k(x)(x - \bar{x})^k$$

for a remainder (error) function $h_k(x)$ such that

$$\lim_{x \to \bar{x}} h_k(x) = 0$$

- the error of the approximation in the vicinity of $\bar{x}$ is thus smaller than order $(x - \bar{x})^k$

For the 'first-order' case ($k = 1$)

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + h_1(x)(x - \bar{x}) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + o(x - \bar{x})$$

Alternative approaches that formalize the idea of the perturbation method in dynamic settings.

- here, we rely on the series expansion method (Holmes (1995), Lombardo (2010), Borovička and Hansen (2014))

Let x_t be a Markov stochastic process of the form

$$x_{t+1} = \psi(x_t, w_{t+1}) \qquad w_{t+1} \sim N(0, I)$$

- ψ is a nonlinear function

Consider the class of models indexed by a perturbation parameter q

$$x_{t+1}(q) = \psi(x_t(q), q w_{t+1}, q) \tag{3.12}$$

- the perturbation parameter q scales the volatility of the Gaussian shock
- each q implies a different process $x_t(q)$

Class of models indexed by a perturbation parameter q

$$x_{t+1}(q) = \psi\left(x_t(q), q w_{t+1}, q\right)$$

For $q = 0$, we obtain the deterministic model

$$x_{t+1}(0) = \psi\left(x_t(0), 0, 0\right)$$

- assume that there exists a fixed point $\bar{x}$ that solves this deterministic equation, called the steady state:

  $$\bar{x} = \psi\left(\bar{x}, 0, 0\right)$$

  such that $x_t(0) = \bar{x}$.

For $q = 1$, we recover the original model

$$x_{t+1}(1) = \psi\left(x_t(1), w_{t+1}, 1\right)$$

Assume that there exists a series expansion of the process x_t around $q = 0$

$$x_t(q) \approx \bar{x} + q x_{1t} + \frac{q^2}{2} x_{2t} + \ldots$$

- this extends the idea of Taylor expansion to a stochastic environment

The processes x_{jt} can be viewed as derivatives of x_t with respect to the perturbation parameter.

- their laws of motion can be inferred by differentiating the law of motion (3.12) j times and evaluating the derivatives at $q = 0$

$$\begin{aligned} \bar{x} &= \psi(\bar{x}, 0, 0) \\ x_{1t+1} &= \psi_q + \psi_x x_{1t} + \psi_w w_{t+1} \end{aligned}$$

where

$$\psi_q = \frac{\partial \psi}{\partial q}(\bar{x}, 0, 0) \qquad \psi_x = \frac{\partial \psi}{\partial x}(\bar{x}, 0, 0) \qquad \psi_w = \frac{\partial \psi}{\partial w}(\bar{x}, 0, 0)$$

and so on for higher orders

Asset pricing

- Campbell and Shiller (1988) approximation of returns dynamics
- log-linear approximation of the price-dividend ratio

Optimal control problems in macroeconomics

- approximation of a control problem using a model with linear constraints and a quadratic objective function

Equilibrium models

- solving for linearized versions of equilibrium models

We apply the series expansion method to approximate dynamics of asset returns.

Start with the definition of the return

$$R_{t+1} = \frac{Q_{t+1} + G_{t+1}}{Q_t} = \frac{Q_{t+1}/G_{t+1} + 1}{Q_t/G_t} \frac{G_{t+1}}{G_t}$$

and rewrite it in logarithms of the above quantities

$$r_{t+1} = \log R_{t+1} \qquad q_t = \log \frac{Q_t}{G_t} \qquad g_{t+1} = \log \frac{G_{t+1}}{G_t}$$

to obtain

$$\exp(r_{t+1}) = \frac{\exp(q_{t+1}) + 1}{\exp(q_t)} \exp(g_{t+1})$$

Now use the first-order series expansion

$$\begin{aligned} r_{t+1} &\approx \bar{r} + \mathsf{q} r_{1t+1} \\ q_t &\approx \bar{q} + \mathsf{q} q_{1t} \\ g_{t+1} &\approx \bar{g} + \mathsf{q} g_{1t+1} \end{aligned}$$

We thus obtain the expression for the asset return (written in logarithms)

$$\bar{r} + \mathsf{q} r_{1t+1} = \log\left[\exp\left(\bar{q} + \mathsf{q} q_{1t+1}\right) + 1\right] - \left(\bar{q} + \mathsf{q} q_{1t}\right) + \left(\bar{g} + \mathsf{q} g_{1t+1}\right)$$

Apply the series expansion:

- evaluate the return equation at $\mathsf{q} = 0$:

$$\bar{r} = \log\left[\exp\left(\bar{q}\right) + 1\right] - \bar{q} + \bar{g}$$

- differentiate with respect to q and evaluate the derivative at $\mathsf{q} = 0$:

$$r_{1t+1} = \frac{\exp\left(\bar{q}\right)}{\exp\left(\bar{q}\right) + 1} q_{1t+1} - q_{1t} + g_{1t+1}$$

Why is the loglinear approximation useful? We can solve the linear equation forward.

- denote $\rho = \frac{\exp(\bar{q})}{\exp(\bar{q})+1}$, express q_{1t}, and iterate forward

$$\begin{aligned} q_{1t} &= g_{1t+1} - r_{1t+1} + \rho q_{1t+1} = g_{1t+1} - r_{1t+1} + \rho\left(g_{1t+2} - r_{1t+2}\right) + \rho^2 q_{1t+2} = \ldots \\ &= \lim_{T\to\infty} \sum_{j=1}^{T} \rho^j \left(g_{1t+j} - r_{1t+j}\right) + \underbrace{\rho^T q_{1T}}_{\to 0} \end{aligned}$$

We thus obtain

$$q_{1t} = \sum_{j=1}^{\infty} \rho^j \left(g_{1t+j} - r_{1t+j}\right) = \sum_{j=1}^{\infty} \rho^j g_{1t+j} - \sum_{j=1}^{\infty} \rho^j r_{1t+j}$$

- this is an accounting identity, which follows solely from the definition of the return (must always hold, does not assume any particular model
- when the price-dividend ratio is high today (relative to $\bar{q}$), then either future dividend growth must be high, or future returns must be high

We derive a similar equation from a different perspective. Consider the valuation equation

$$\frac{Q_t}{G_t} = E_t\left[\frac{S_{t+1}}{S_t}\frac{G_{t+1}}{G_t}\left(\frac{Q_{t+1}}{G_{t+1}} + 1\right)\right]$$

- denote

$$q_t = \log\frac{Q_t}{G_t} \qquad s_{t+1} = \log\frac{S_{t+1}}{S_t} \qquad g_{t+1} = \log\frac{G_{t+1}}{G_t}$$

  then

$$\exp(q_t) = E_t\left[\exp(s_{t+1} + g_{t+1})\left(\exp(q_{t+1}) + 1\right)\right]$$

Now assume that each of the processes q_t, s_t, g_t can be written in the series expansion form (for perturbation parameter q)

$$\exp(\bar{q} + \mathrm{q} q_{1t}) = E_t\left[\exp(\bar{s} + \mathrm{q} s_{1t+1} + \bar{g} + \mathrm{q} g_{1t+1})\left(\exp(\bar{q} + \mathrm{q} q_{1t+1}) + 1\right)\right]$$

Differentiate

$$\exp\left(\bar{q} + \mathsf{q} q_{1t}\right) = E_t\left[\exp\left(\bar{s} + \mathsf{q} s_{1t+1} + \bar{g} + \mathsf{q} g_{1t+1}\right)\left(\exp\left(\bar{q} + \mathsf{q} q_{1t+1}\right) + 1\right)\right]$$

with respect to q to obtain

$$\begin{aligned} \exp\left(\bar{q}\right) &= \exp\left(\bar{s} + \bar{g}\right)\left(\exp\left(\bar{q}\right) + 1\right) \\ \exp\left(\bar{q}\right) q_{1t} &= E_t\left[\exp\left(\bar{s} + \bar{g}\right)\left(s_{1t+1} + g_{1t+1}\right)\left(\exp\left(\bar{q}\right) + 1\right) + \exp\left(\bar{s} + \bar{g}\right)\exp\left(\bar{q}\right) q_{1t+1}\right] \end{aligned}$$

The latter equation can be rewritten as

$$q_{1t} = E_t\left[\frac{\exp\left(\bar{s} + \bar{g}\right)\left(\exp\left(\bar{q}\right) + 1\right)}{\exp\left(\bar{q}\right)}\left(s_{1t+1} + g_{1t+1}\right) + \exp\left(\bar{s} + \bar{g}\right) q_{1t+1}\right]$$

Using the steady-state equation, we get

$$q_{1t} = E_t\left[s_{1t+1} + g_{1t+1} + \exp\left(\bar{s} + \bar{g}\right) q_{1t+1}\right].$$

Impose linear dynamics on the model

- linear law of motion for the state $x_t \in \mathbb{R}^n$

$$x_{t+1} = A_o x_t + C w_{t+1}, \qquad w_{t+1} \sim N(0, I_p)$$

- linear structure of the SDF and dividend growth rate

$$\begin{aligned} s_{1t+1} &= S x_{t+1} \\ g_{1t+1} &= G x_{t+1} \end{aligned}$$

  where S and G are $1 \times n$ vectors

Conjecture that the solution for the price-dividend ratio is also linear:

$$q_{1t} = Q x_t$$

The valuation equation implies

$$\begin{aligned} Qx_t &= E_t \left[Sx_{t+1} + Gx_{t+1} + \exp\left(\bar{s} + \bar{g}\right) Qx_{t+1} \right] \\ &= \left(S + G\right) A_o x_t + \exp\left(\bar{s} + \bar{g}\right) QA_o x_t \end{aligned}$$

This equation has to hold for every value of x_t, so coefficients have to match

$$Q = \left(S + G\right) A_o + \exp\left(\bar{s} + \bar{g}\right) QA_o$$

We can therefore solve for the vector Q

$$Q = \left(S + G\right) A_o \left[I - \exp\left(\bar{s} + \bar{g}\right) A_o \right]^{-1}$$

Solving for the price dividend ratio

What are we losing with the linear approximation

$$q_{1t} = Qx_t = (S + G)\, A_o \left[I - \exp\left(\bar{s} + \bar{g}\right) A_o\right]^{-1} x_t$$

- the mapping between state and the price-dividend ratio does not depend on uncertainty C

Recall that risk premia are given by covariances of the stochastic discount factor with returns:

$$0 = E_t\left[\frac{S_{t+1}}{S_t}\left(R_{t+1} - R^f_{t+1}\right)\right] \implies E_t\left[R_{t+1} - R^f_{t+1}\right] = -R^f_{t+1} Cov_t\left[\frac{S_{t+1}}{S_t}, R_{t+1} - R^f_{t+1}\right]$$

The linear approximation

$$q_{1t} = E_t\left[s_{1t+1} + g_{1t+1} + \exp\left(\bar{s} + \bar{g}\right) q_{1t+1}\right].$$

neglects these covariances.

- risk premia in a smooth model of preferences are a 'second-order' concept

$$\exp\left(\bar{q} + \mathsf{q} q_{1t}\right) = E_t\left[\exp\left(\bar{s} + \mathsf{q} s_{1t+1} + \bar{g} + \mathsf{q} g_{1t+1}\right)\left(\exp\left(\bar{q} + \mathsf{q} q_{1t+1}\right) + 1\right)\right]$$

Let us first manipulate the expression on the right-hand side. Substitute

$$s_{1t+1} = Sx_{t+1} \qquad g_{1t+1} = Gx_{t+1} \qquad x_{t+1} = A_o x_t + Cw_{t+1}$$

to obtain

$$\begin{aligned} \exp\left(\bar{q} + \mathsf{q} q_{1t}\right) &= E_t\left[\exp\left(\bar{s} + \bar{g} + \bar{q} + \mathsf{q}\left(S + G + Q\right)\left(A_o x_t + Cw_{t+1}\right)\right)\right] \\ &\quad + E_t\left[\exp\left(\bar{s} + \bar{g} + \mathsf{q}\left(S + G\right)\left(A_o x_t + Cw_{t+1}\right)\right)\right] \end{aligned}$$

Collect the deterministic and random components

$$\begin{aligned} \exp\left(\bar{q} + \mathsf{q} q_{1t}\right) &= \exp\left(\bar{s} + \bar{g} + \bar{q} + \mathsf{q}\left(S + G + Q\right) A_o x_t\right) E_t\left[\exp\left(\mathsf{q}\left(S + G + Q\right) Cw_{t+1}\right)\right] \\ &\quad + \exp\left(\bar{s} + \bar{g} + \mathsf{q}\left(S + G\right) A_o x_t\right) E_t\left[\exp\left(\mathsf{q}\left(S + G\right) Cw_{t+1}\right)\right] \end{aligned}$$

Now utilize an expression for the expectation of a log-normally distributed random variable

$$w \sim N(0, I) \qquad \Longrightarrow \qquad E\left[\exp\left(\mu + \sigma w\right)\right] = \exp\left(\mu + \frac{1}{2}\sigma\sigma'\right)$$

Here, $\sigma = \mathsf{q}\left(S + G + Q\right)$ and $\sigma = \mathsf{q}\left(S + G\right)$

$$\begin{aligned} \exp\left(\bar{q} + \mathsf{q} q_{1t}\right) &= \exp\left(\bar{s} + \bar{g} + \bar{q} + \mathsf{q}\left(S + G + Q\right) A_o x_t + \frac{1}{2}\mathsf{q}^2 \left(S + G + Q\right) CC' \left(S + G + Q\right)'\right) \\ &\quad + \exp\left(\bar{s} + \bar{g} + \mathsf{q}\left(S + G\right) A_o x_t + \frac{1}{2}\mathsf{q}^2 \left(S + G\right) CC' \left(S + G\right)'\right) \end{aligned}$$

The effect of risk premia is embedded in

$$\frac{1}{2}\mathsf{q}^2 \left(S + G\right) CC' \left(S + G\right)'$$

- interaction of uncertainty in the SDF and cash flows
- depends on the uncertainty of the underlying economy in CC'

Risk-premium contribution

$$\frac{1}{2}q^2\,(S+G)\,CC'\,(S+G)'$$

The risk premium scales with q^2, so it vanishes in the linear approximation.

- in the perturbation, terms with q^2 are higher order relative to terms with q

Solutions

- higher-order approximation
- a different type of series expansion (Borovička and Hansen (2014), Bhandari et al. (2019))

Consider the following decision problem in sequence formulation:

$$\max_{\{a_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(x_t, a_t\right) \qquad \text{subject to } x_{t+1} = \psi\left(x_t, a_t, w_{t+1}\right)$$

- x_t is the state vector, with x_0 given
- a_t is the vector of controls that affect utility and the evolution of the state
- ψ is the set of restrictions that determine the controlled law of motion for the state

We can apply the same perturbation logic here.

- convenient choice: second-order expansion of the utility function and first-order expansion of the law of motion

Approximate the state and control dynamics to first order

$$x_t \approx \bar{x} + q x_{1t} \qquad a_t = \bar{a} + q a_{1t}$$

Approximate the law of motion to first order

$$\begin{aligned} \bar{x} &= \psi\left(\bar{x}, \bar{a}, 0\right) \\ x_{1t+1} &= \psi_q + \psi_x x_{1t} + \psi_a a_{1t} + \psi_w w_{t+1} \end{aligned}$$

- partial derivatives ψ_q, ψ_x and ψ_w evaluated at the steady state $(\bar{x}, \bar{a}, 0)$.

Approximate the utility function to second order

$$u\left(x_t\left(q\right), a_t\left(q\right), q\right) = u_t\left(q\right) \approx \bar{u} + q u_{1t} + \frac{q^2}{2} u_{2t} \tag{3.13}$$

To obtain $\bar{u}$, u_{1t}, and u_{2t}, evaluate the derivatives of

$$u\left(x_t\left(q\right), a_t\left(q\right)\right) \approx u\left(\bar{x} + q x_{1t}, \bar{a} + q a_{1t}, q\right)$$

Zero-th order derivative of the utility function

$$\bar{u} = u\left(\bar{x}, \bar{a}, 0\right).$$

First order derivative of the utility function

$$u_{1t} = u_q + u_x x_{1t} + u_a a_{1t}$$

Second order derivative

$$u_{2t} = u_{qq} + 2u_{qx}x_{1t} + 2u_{qa}a_{1t} + x_{1t}'u_{xx}x_{1t} + a_{1t}'u_{aa}a_{1t} + 2a_{1t}'u_{ax}x_{1t}$$

where all partial derivatives of u are evaluated at the steady state $(\bar{x}, \bar{a}, 0)$

Then we can construct the second-order approximation of u_t in (3.13) evaluated at q $= 1$, by combining $\bar{u}$, u_{1t} and $\frac{1}{2}u_{2t}$.

We thus obtain the decision problem

$$\max_{\{a_{1t}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \hat{u}(x_t, a_t)$$

with

$$\hat{u}(x_t, a_t) = \bar{u} + u_q + \frac{1}{2} u_{qq} + (u_x + u_{qx}) x_{1t} + (u_a + u_{qa}) a_{1t} + \frac{1}{2} x_{1t}' u_{xx} x_{1t} + \frac{1}{2} a_{1t}' u_{aa} a_{1t} + a_{1t}' u_{ax} x_{1t}$$

subject to

$$x_{1t+1} = \psi_q + \psi_x x_{1t} + \psi_a a_{1t} + \psi_w w_{t+1}$$

with x_0 given.

This is a linear-quadratic problem with a tractable solution even for high-dimensional state spaces.

- solution based on computation of quadratic sums using formula (3.10) together with an optimization step
- the crucial observation is that the optimal policy a_t^* is linear in the state x_t

Dynamic equilibria in macroeconomics often feature a combination of backward-looking and forward-looking equations.

- a backward-looking equation represents the current value of a variable as a function of past values

$$x_{t+1} = A_o x_t + C w_{t+1}$$

- a forward-looking equation represents the current value of a variable as a function of future values

$$q_{1t} = E_t \left[s_{1t+1} + g_{1t+1} + \exp\left(\bar{s} + \bar{g} \right) q_{1t+1} \right]$$

We want to find a solution for all involved variables:

- backward-looking evolution of an appropriately defined state
- a mapping from the state to all remaining endogenous variables

In the case of the price-dividend ratio, we imposed

$$x_{t+1} = A_o x_t + C w_{t+1} \qquad s_{1t+1} = S x_{t+1} \qquad g_{1t+1} = G x_{t+1}$$

and solved the forward-looking equation

$$q_{1t} = E_t \left[s_{1t+1} + g_{1t+1} + \exp\left(\bar{s} + \bar{g}\right) q_{1t+1} \right]$$

for $q_{1t} = Q x_t$.

Another example: neoclassical growth model

- backward-looking equation: law of motion for capital

$$k_{t+1} = (1-\delta)\,k_t + G\,(k_t) - c_t$$

- forward-looking equation: optimal consumption choice

$$U'\,(c_t) = \beta U'\,(c_{t+1})\left(1-\delta+G'\,(k_{t+1})\right)$$

Solution involves finding the mapping $c_t = c\,(k_t)$.

There are well-established methods for solving these sets of forward- and backward-looking equations

- Ljungqvist and Sargent (2020), Chapter 6, Blanchard and Kahn (1980), Sims (2002)
- implementation in available packages, for example Dynare

VAR estimation

Another advantage of the linear system lies in estimation:

$$x_{t+1} = A_o x_t + C w_{t+1}$$

- estimate rows of A_0 using OLS, equation by equation
- collect residuals from all equations, compute covariance to obtain CC'

Covariance matrix CC' cannot be used to find a unique C

- this is important for shock identification, when we want to know the impact of individual components of w_{t+1}

A_o and CC' may have a particular structure, for example implied by a macroeconomic model

- these cross-equation restrictions must be incorporated in estimation
- have to use GMM, maximum likelihood or Bayesian estimation instead of OLS

Summary

Summary

Linear state space models are widely used for their tractability.

- they can handle multidimensional state spaces at negligible computational costs
- they cannot handle nonlinearities

Perturbation methods provide linear approximation of nonlinear models

- they are an example of local approximations: methods work well in the neighborhood of a particular point in the state space, and become less accurate further away
- asset pricing: loglinearization of price-dividend ratios (Campbell and Shiller (1988))
- linear-quadratic dynamic programming: Ljungqvist and Sargent (2020), Chapters 3, 5, and 6
- linear solutions of equilibrium (DSGE) models: Blanchard and Kahn (1980), Sims (2002)
- methods can be extended to higher-order perturbations (Judd (1998), Chapter 13)

Appendix

Bhandari, Anmol, Jaroslav Borovička, and Paul Ho (2019) "Survey Data and Subjective Beliefs in Business Cycle Models," Federal Reserve Bank of Richmond Working Paper No. 19-14.

Blanchard, Olivier Jean and Charles M. Kahn (1980) "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48 (5), 1305–1312.

Borovička, Jaroslav and Lars Peter Hansen (2014) "Examining Macroeconomic Models through the Lens of Asset Pricing," *Journal of Econometrics*, 183 (1), 67–90.

Campbell, John Y. and Robert J. Shiller (1988) "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies*, 1, 195–228.

Frisch, Ragnar (1933) "Propagation Problems and Impulse Problems in Dynamic Economics," in *Economic Essays in Honour of Gustav Cassel*, 171–205: Allen and Unwin.

Holmes, Mark H. (1995) *Introduction to Perturbation Methods*: Springer.

Judd, Kenneth L. (1998) *Numerical Methods in Economics*: The MIT Press, Cambridge, MA.

Kydland, Finn E. and Edward C. Prescott (1982) "Time to Build and Aggregate Fluctuations," *Econometrica*, 50 (6), 1345–1370.

Ljungqvist, Lars and Thomas J. Sargent (2020) "Recursive Macroeconomic Theory," Unpublished manuscript, draft of 5th edition.

Lombardo, Giovanni (2010) "On Approximating DSGE Models by Series Expansions," ECB Working paper No. 1264.

Samuelson, Paul A. (1939) "Interactions between the Multiplier Analysis and the Principle of Acceleration," *Review of Economics and Statistics*, 21 (2), 75–78.

Sims, Christopher A. (2002) "Solving Rational Expectations Models," *Computational Economics*, 20 (1–2), 1–20.

Slutsky, Eugen (1927) "The Summation of Random Causes as the Source of Cyclic Processes," in *Problems of Economic Conditions*, 3, Chap. 1: The Conjuncture Institute, Moscow.

——— (1937) "The Summation of Random Causes as the Source of Cyclic Processes," *Econometrica*, 5 (2), 105–146.

Yule, George Udny (1927) "On a Method of Investigating Periodicities in Disturbed Series, with Special Reference to Wolfer's Sunspot Numbers," *Philosophical Transactions of the Royal Society of London. Series A*, 226 (636–646), 267–298.