ECON-GA 4091/GB 5091 — Computational Dynamics
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Problem Set 2

Due on Friday, February 10, 11 am, via Brightspace

For each of the problems, submit your answers and code that generates the results. A Jupyter notebook is fine, as long as the results and code are well organized, with appropriate discussion. Group discussion is encouraged but everybody has to produce code and writeup individually. You can use and adapt the available codes from Github if you wish.

1 Mean-preserving spread in the McCall (1970) model

We study the comparative statics in the McCall (1970) model, given by the Bellman equation

$$V(w) = \max_{\{\text{accept, reject}\}} \left\{ \frac{w}{1-\beta}, c + \beta \int_0^B V(w')\, dF(w') \right\}.$$

Let $F(w)$ be the cdf of a random variable W with Beta distribution with parameters α and β

$$W \sim Beta(\alpha_w, \beta_w).$$

We study the sensitivity of the reservation wage $\bar{w}$ with respect to changes of the parameters α_w and β_w. The mean and variance of the random variable W with *Beta* distribution are

$$\begin{aligned} E[W|\alpha_w, \beta_w] &= \frac{\alpha_w}{\alpha_w + \beta_w} \\ Var[W|\alpha_w, \beta_w] &= \frac{\alpha_w \beta_w}{(\alpha_w + \beta_w)^2 (\alpha_w + \beta_w + 1)} \end{aligned}$$

Choose parameters $\beta = 0.96$, $c = 0.2$, $B = 1$.

Question 1.1 Start with $\alpha_w = \beta_w = 10$. Vary the parameter α_w on a suitably chosen grid of parameter values α_w^i, $i = 1, \ldots, I$, from 10 down to 0.1, with $\alpha_w^1 = 10$ and $\alpha_w^I = 0.1$. For each value of α_w^i, pick β_w^i so that $E\left[W|\alpha_w^i, \beta_w^i\right]$ remains unchanged.

For each pair $\left(\alpha_w^i, \beta_w^i\right)$, compute $Var\left[W|\alpha_w^i, \beta_w^i\right]$ and show that $Var\left[W|\alpha_w^i, \beta_w^i\right]$ increases with i.

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Question 1.2 For each pair $\left(\alpha_w^i, \beta_w^i\right)$, use your preferred method to solve for the reservation wage, denoted $\bar{w}^i$. Plot the resulting reservation wage as a function of $Var\left[W|\alpha_w^i, \beta_w^i\right]$. Provide an economic explanation for the shape of the relationship.

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2 Monte Carlo integration in the McCall (1970) model

The reservation wage $\bar{w}$ in the McCall (1970) model can be found by solving the fixed point problem

$$Q = c + \beta \int_0^B \max_{\{\text{accept, reject}\}} \left\{ \frac{w'}{1-\beta}, Q \right\} dF\left(w'\right)$$

for Q, which is the value of rejecting an offer. Optimal choice of the reservation wage $\bar{w}$ then also implies

$$Q = \frac{\bar{w}}{1-\beta}.$$

We use the contraction mapping argument to implement the successive approximation scheme

$$Q_{n+1} = c + \beta \int_0^B \max_{\{\text{accept, reject}\}} \left\{ \frac{w'}{1-\beta}, Q_n \right\} dF\left(w'\right)$$

starting from an initial guess Q_0.

In this problem, we study the Monte Carlo approach to the evaluation of the integral in the Bellman equation. The Monte Carlo approach approximates the expectation of a function $g\left(W\right)$ of a random variable W using the empirical distribution constructed using random draws w^i, $i = 1, \ldots, I$ from the distribution of W:

$$E\left[g\left(W\right)\right] = \int g\left(w\right) dF\left(w\right) \approx \sum_{i=1}^{I} g\left(w^i\right).$$

The law of large numbers implies, under appropriate conditions, that the right-hand side converges to the expectation on the left-hand side as $I \to \infty$.

Choose parameters $\beta = 0.96$, $c = 0.2$, $B = 1$. The function $F\left(w\right)$ is the cdf of a *Beta*-distributed random variable

$$W \sim Beta\left(\alpha_w, \beta_w\right)$$

with $\alpha_w = 0.5$ and $\beta_w = 0.5$.

Question 2.1 Start from the initial guess $Q_0 = \frac{1}{2}\left(1-\beta\right)^{-1} B$. Every iteration n, draw a random sample of $I = 100$ realizations of the random variable W, denoted w^i. Evaluate

$$\int_0^B \max_{\{\text{accept, reject}\}} \left\{ \frac{w'}{1-\beta}, Q_n \right\} dF\left(w'\right) \approx \sum_{i=1}^{I} \max_{\{\text{accept, reject}\}} \left\{ \frac{w^i}{1-\beta}, Q_n \right\}$$

and compute Q_{n+1}. Iterate for $n = 0, 1, \ldots, N$, where $N = 200$.

Notice that in every period, you draw a new random sample.

Convert every Q_n to reservation wage $\bar{w}_n = \left(1-\beta\right) Q_n$, and plot $\bar{w}_n$ as a function of n.

Compare the path $\bar{w}_n$ with the 'true' value $\bar{w}$ computed using a different method (bisection, Newton–Raphson, ...).

Does $\bar{w}_n$ converge to $\bar{w}$ as $n \to \infty$? Explain.

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Question 2.2 Would convergence issues from the previous question be resolved if we draw one random sample of I draws from the distribution $F(w)$ at the beginning, and then use the same draws in every iteration n? Explain. If not, do you have a different proposal how to make sure Monte Carlo leads to a reliable estimate of $\bar{w}$?

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3 Modifications of the McCall (1970) model

In thie problem, we consider two modifications of the McCall (1970) problem. The baseline problem is characterized by the Bellman equation

$$V(w) = \max_{\{\text{accept, reject}\}} \left\{ \frac{w}{1-\beta}, c + \beta \int_0^B V(w')\, dF(w') \right\}$$

that yields optimal policy in the form of a reservation wage $\bar{w}$.

Choose baseline parameters $\beta = 0.96$, $c = 0.4$, $B = 1$, and cdf $F(w)$ given by a *Beta* distribution

$$W \sim Beta(\alpha_w, \beta_w)$$

with $\alpha_w = \beta_w = 0.5$.

Question 3.1 Modify the baseline problem as follows. After receiving wage offer w, the worker has three options. A) accept the offer w for permanent employment, in which case the worker leaves the labor market and works at wage w forever, B) accept the offer w for temporary employment, in which case the worker works at w only in the current period but rejoins search at the beginning of next period, C) reject the offer.

Characterize the reservation wages $\bar{w}_P$ and $\bar{w}_T$ above which the worker accepts permanent and temporary employment, respectively. Solve the problem numerically using your preferred method, compare the reservation wages to the reservation wage $\bar{w}$ in the baseline model, and show a graph displaying the value functions in baseline and modified models. Explain the differences.

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Question 3.2 There is an exogenously specified two-state Markov chain x_t with transition matrix $\mathbf{P}$ that determines the distribution of offers in period t. In particular, when $x_t = e_i$, $i = 1, 2$, then w_t is drawn from distribution $Beta\left(\alpha_w^i, \beta_w^i\right)$. Verify that the optimal policy takes the form of a state-dependent reservation wage $\bar{w}_i$. Use offer distribution parameters

$\alpha_w^1 = \beta_w^1 = 0.5$ and $\alpha_w^2 = \alpha_w^2 = 2$, and transition matrix

$$\mathbf{P} = \begin{pmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{pmatrix}$$

with $\phi = 0.8$. Solve the problem using your preferred method, compare the reservation wages $\bar{w}_i$ with the reservation wage in the baseline problem $\bar{w}$, and show a graph displaying the state-dependent value function $V_i(w)$, $i = 1, 2$ together with the baseline value function $V(w)$. Explain the differences.

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References

McCall, John (1970) "Economics of Information and Job Search," *Quarterly Journal of Economics*, 84 (1), 113–126.