ECON-GA 4091/GB 5091 — Computational Dynamics
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Problem Set 4

Due on Friday, February 24, 11 am, via Brightspace

For each of the problems, submit your answers and code that generates the results. A Jupyter notebook is fine, as long as the results and code are well organized, with appropriate discussion. Group discussion is encouraged but everybody has to produce code and writeup individually. You can use and adapt the available codes from Github if you wish.

1 Estimation of a time path using Kalman filter

We are given the state space model

$$\begin{array}{rclll} x_{t+1} & = & A_o x_t + C w_{t+1} & n \times 1 & w_{t+1} \sim N\left(0, I_p\right) \\ y_t & = & G x_t + v_t & m \times 1 & v_t \sim N\left(0, R\right). \end{array} \tag{1.1}$$

Here, x_t is an unobservable state vector that follows a Gaussian Markov process, y_t is a measurement vector and v_t is measurement noise independent of $\{w_t\}_{t=1}^{\infty}$. We assume an initial condition

$$x_0 \sim N\left(\hat{x}_0, \Sigma_0\right). \tag{1.2}$$

The Kalman (1960) filtering algorithm leads to a recursive formula for the best predictor of x_t given observations $y^{t-1} = (y_{t-1}, \ldots, y_0)$. Because of the linear-Gaussian structure of the whole system, we can infer that the predictor will also be Gaussian, so it is sufficient to construct predictors for the first two moments of its distribution

$$\begin{array}{rcl} \hat{x}_t & = & E\left[x_t \mid y^{t-1}\right] \\ \Sigma_t & = & E\left[\left(x_t - \hat{x}_t\right)\left(x_t - \hat{x}_t\right)'\right]. \end{array} \tag{1.3}$$

Question 1.1 Write a function that implements the Kalman filtering formula. It takes as input the parameters of the model, the parameters of the prior, and the data for a given sequence of observations $y_0, \ldots y_T$, and produces as output the corresponding paths of $\hat{x}_t$ and Σ_t. You do not need to re-derive the formulas.

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Question 1.2 Assume that the underlying state x_t is scalar, and we have a scalar observable variable y_t. Download quarterly data on real GDP (FRED series GDPC1), and compute the quarterly annualized (logarithmic) growth rate

$$g_t = 4 \log \frac{Y_t}{Y_{t-1}}.$$

Demean the time series. Use it as an input y_t for the Kalman filter, and produce the resulting time series for the filtered mean of the state $\hat{x}_t$ and posterior variance Σ_t. Use the prior $x_0 = 0$, $\Sigma_0 = 0.01$. Use the following parameter values for the state space system:

1. Benchmark parameterization: $A_o = 0.95$, $C = 0.02$, $G = 1$, $R = 0.05^2$.
2. Less persistent state: Relative to the benchmark parameterization, set $\widetilde{A}_o = 0.95$ and adjusts C so that the unconditional variance of the state remains unchanged. Recall that the unconditional variance $Var(x_t)$ is given by the equation

$$Var(x_t) = A_o Var(x_t) A_0' + CC'.$$

3. More noise in the signal: Relative to the benchmark parameterization, set $\widetilde{R} = 0.15^2$.

Produce a plot with the demeaned GDP growth data and the filtered path $\hat{x}_t$ for all three cases. Produce a plot with the evolution of the covariance Σ_t. Provide a discussion of the results, specifically focusing on the differences between the filtered paths and evolution of the covariance for the three parameterizations.

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2 Filtering Bernoulli draws

In this problem, we derive the optimal filter for the parameter of a Bernoulli distribution. Let y_t, $t = 1, 2, \ldots$ be iid draws from a Bernoulli distribution with parameter θ, meaning that $y_t = 1$ with probability θ and $y_t = 0$ with probability $1 - \theta$.

The agent does not know θ but updates beliefs about θ based on observed realizations of y_t.

Assume that the prior $p(\theta)$ that the agent has is Beta distributed with positive parameters α_0 and β_0

$$\theta \sim Beta(\alpha_0, \beta_0).$$

This prior implies the density $p(\theta)$ given by

$$p(\theta) = \frac{\theta^{\alpha_0 - 1}(1-\theta)^{\beta_0 - 1}}{B(\alpha_0, \beta_0)} \qquad \theta \in (0, 1).$$

For example, for $\alpha_0 = \beta_0 = 1$, we obtain a uniform prior.

Question 2.1 Using Bayes law, derive a recursive formula for the posterior $p\left(\theta | y^t\right)$ where $y^t = (y_1, \ldots, y_t)$. Specifically, show that if the posterior $p\left(\theta | y^{t-1}\right)$ is Beta distributed with parameters α_{t-1} and β_{t-1}, then $p\left(\theta | y^t\right)$ is also Beta distributed with parameters α_t and β_t, and derive a recursive formula for α_t and β_t.

Hint: The Bayes formula combined with the fact that the draws are iid allows you to write (convince yourself that this is indeed the case)

$$p\left(\theta|y^{t}\right)=\frac{p\left(y_{t}|\theta\right)p\left(\theta|y^{t-1}\right)}{p\left(y_{t}|y^{t-1}\right)}.$$

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Question 2.2 Imagine, for simplicity, that occurrence of a recession in a given time period is iid over time. Denote $y_t = 1$ if the economy is in a recession in period t. Go to the NBER business cycle dates webpage

https://www.nber.org/research/data/us-business-cycle-expansions-and-contractions

and encode all quarters t since 1854 during which the economy was in a recession as $y_t = 1$, and $y_t = 0$ otherwise. A recession is the time span that starts one quarter after the peak quarter, and ends in the quarter denoted as trough quarter (alternatively, you can also do it for months).

Denote θ the probability that the economy is in a recession in a given period. Let the prior $p(\theta)$ be uniform on $[0,1]$, i.e., Beta distributed with parameters $\alpha_0 = \beta_0 = 1$. Feed in the recession and expansion data starting from 1855 and plot the evolution of the posterior mean $E\left[\theta|y^t\right]$ and posterior variance $Var\left[\theta|y^t\right]$ for $y = 1, 2, \ldots$.

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References

Kalman, Rudolph Emil (1960) "New Approach to Linear Filtering and Prediction Problems," *Transactions of the ASME–Journal of Basic Engineering*, 82 (Series D), 35–45.