Yannick Dillschneider and Raimond Maurer

Functional Ross Recovery: Theoretical Results and Empirical Tests

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WHAT IS THE RECOVERY PROBLEM?

Using observed cross-section(s) of prices (of Arrow–Debreu securities), infer

- preference parameters
- investors' beliefs

imposing 'as little structure as possible'. Only:

- Markovianity
- time invariance
- minimal restrictions on preferences

This is an identification problem.

A FINITE STATE SPACE FRAMEWORK

Physical environment

- X — a discrete-time stationary and ergodic Markov chain with n states

Investor beliefs and preferences

- $\mathbf{P} = [p_{ij}]$ — transition matrix — investors' beliefs (here identical to DGP)

$$p_{ij} = P\left(X_{t+1} = j \mid X_t = i\right)$$

- $\mathbf{\Psi} = [\psi_{ij}]$ — stochastic discount factor

ψ_{ij} — state-specific discount rate between states i and j

Asset prices

- $\mathbf{V} = [v_{ij}]$ — matrix of prices of one-period Arrow securities

v_{ij} — price in state i of one unit of state-j cash flow next period

AN IDENTIFICATION PROBLEM

Arrow prices encode both beliefs and preferences:

$$v_{ij} = p_{ij}\psi_{ij}$$

Suppose we observe asset prices $[v_{ij}]$.

- Identification problem: Can we separately identify $[p_{ij}]$ and $[\psi_{ij}]$?

$$\underbrace{v_{ij}}_{n \times n \text{ equations}} \quad = \quad \underbrace{p_{ij}}_{n \times (n-1) \text{ unknowns}} \quad \underbrace{\psi_{ij}}_{n \times n \text{ unknowns}}$$

Underidentification!!!

An SDF can be factored as

$$\psi_{ij} = \exp(-\delta) \frac{\phi_i}{\phi_j} m_{ij}$$

- $\mathbf{M} = [m_{ij}]$ forms a martingale, ϕ strictly positive
- consequence of the Perron–Frobenius Theorem
- unique if $\mathbf{V}$ is irreducible (e.g., has strictly positive entries)

Accumulation over time

$$\left[\Psi^{(n)}\right]_{ij} = \underbrace{\exp(-\delta n)}_{\text{long run discount}} \underbrace{\frac{\phi_i}{\phi_j}}_{\text{stationary part}} \underbrace{\left[\mathbf{M}^{(n)}\right]_{ij}}_{\text{martingale}}$$

WHAT IS IDENTIFIED?

$$v_{ij} = \underbrace{\exp(-\delta)\frac{\phi_i}{\phi_j}}_{\substack{\text{identified} \\ \text{from asset} \\ \text{prices}}} \quad \underbrace{m_{ij}p_{ij}}_{\substack{\text{`recovered'} \\ \text{measure } p^{\phi}_{ij} \\ \text{(as a residual)}}}$$

We identified

- long-run discount δ and stationary component ϕ of the SDF
- long-run risk neutral measure $\mathbf{P}^{\phi}$ as a residual

We **did not** identify

- martingale component $\mathbf{M}$ or true belief/DGP $\mathbf{P}$
- since $\mathbf{M}$ can be arbitrary, the cross-section of asset prices on its own does not contain **any** information about beliefs/DGP $\mathbf{P}$

HOW CAN THE LACK OF IDENTIFICATION BE ADDRESSED?

1) Make assumptions on $\mathbf{M}$

- Ross (2015) $\mathbf{M} = \mathbf{1}$
- This assumption then fully identifies $\mathbf{P}$

2) Use time series evidence to identify $\mathbf{P}$ directly

- Conventional time series econometrics approaches
- Caveat: $\mathbf{P}$ represents agents' belief, we need to impose that this belief is correct to identify $\mathbf{P}$ with the DGP.
    - If not, we need additional source of data, like investor surveys

3) Impose more structure on the problem (preferences, functional forms on DGP, model of belief formation, ...)

Researchers typically use combinations of all these approaches.

Theoretical: Generalizations of the recovery results to continuous state spaces

- Perron–Frobenius Theorem for positive matrices generates unique decomposition
- non-uniqueness issues may arise in continuous state spaces

Empirical: Apply the theorem and study the recovered P^{ϕ}

- numerical procedure based on approximation of the state price density in a class of B-spline based finite-rank operators
- smoothing needed due to inherent fragility of the numerical problem
- the finite-rank operators satisfy derived theoretical restrictions
- within this class, the eigenfunction ϕ can be computed analytically

CONTRIBUTION OF THIS PAPER: THEORY

In continuous state spaces, additional restrictions need to be imposed to extract a unique pair (δ, ϕ)

- This problem is distinct from the identification problem above.
- Perhaps more technical in nature?

Hansen and Scheinkman (2009), BHS (2016)

- Acknowledge potential multiplicity but pick a solution that satisfies economically appealing conditions.
- The Markov process X should retain a form of stationarity under the recovered measure.
- Approach motivated by the fact that many valuation operators in the literature lead to multiple solutions of the decomposition.

Other assumptions used in the existing literature

- boundary behavior (Carr and Yu (2012), Dubynskiy and Goldstein (2013)), recurrence (Walden (2016), Park (2016), Qin and Linetsky (2016, 2017))

CONTRIBUTION OF THIS PAPER: THEORY

Dillschneider and Maurer (2019)

- Restrict attention to a class of valuation operators that guarantee a unique decomposition
- Jentzsch theorem as the generalization of Perron–Frobenius to general linear spaces
- Assume valuation operators are compact
- Compactness in a sense means 'close to behaving like on a finite set'

Is this a useful approach?

- Checking compactness is very hard in existing asset pricing models (Borovička, Stachurski (2019))
- But the authors choose to approximate density using a class of functions that satisfy compactness
- Also need to compactify the state space

CONTRIBUTION OF THIS PAPER: EMPIRICS

Paper finds a very convenient approximation technique

- once the B-spline surface is fitted, eigenfunction in closed form

Recovery approach is fragile to discretization and truncation errors

- especially when the underlying Markov process is very persistent
- Walden (2016), Tran and Xia (2014), ...

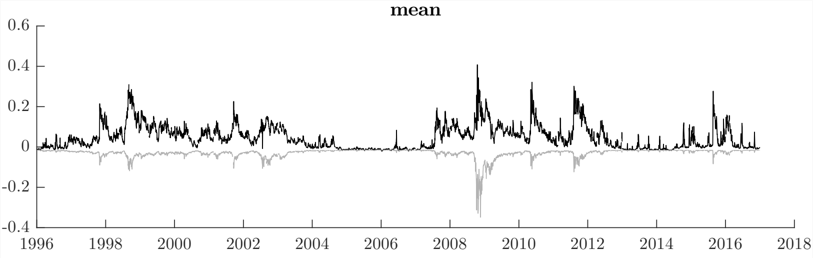
P33: "our functional approach features much fewer degrees of freedom when fitting the state price density"

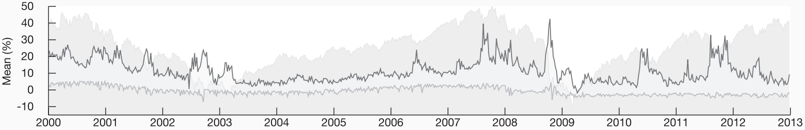
- this is a double-edged sword

It would be useful to have some test examples comparing accuracy of different methods.

- hard to do with actual data, method exhausts overidentifying restrictions

DILLSCHNEIDER AND MAURER (2019) VS AUDRINO ET AL. (2019)





- Recovered moments in black, risk-neutral moments in grey.
- DM top (B-splines), AHL bottom (neural network)

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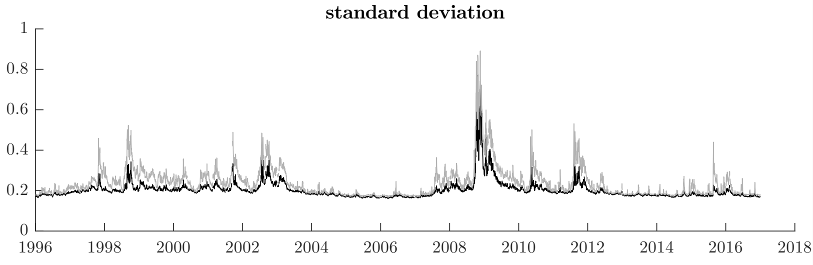

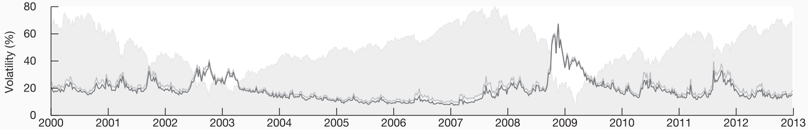

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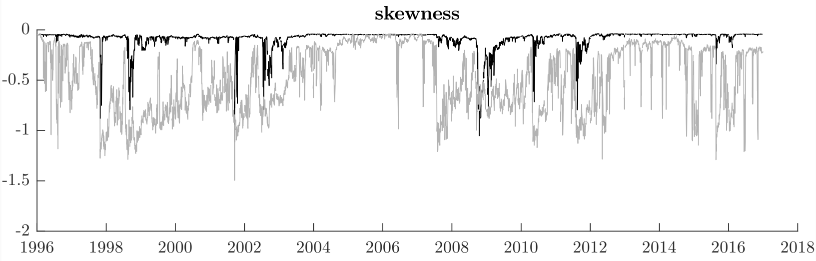

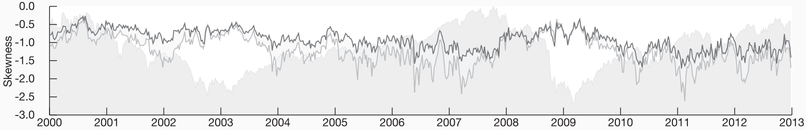

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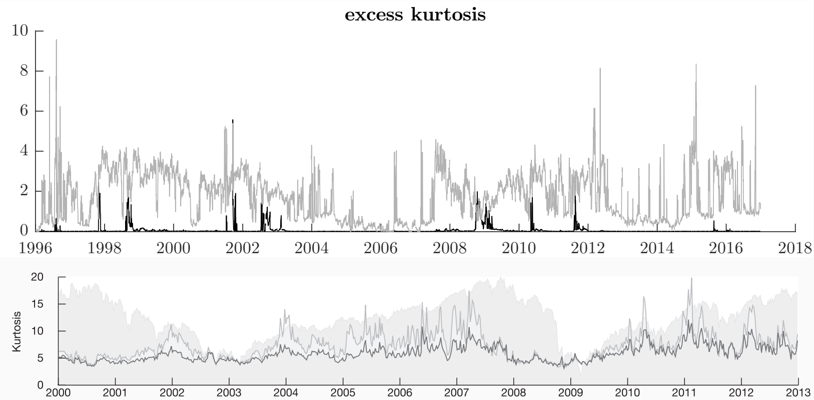

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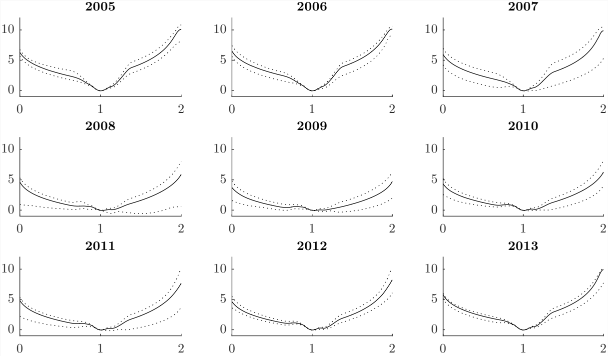
RECOVERED EIGENFUNCTIONS (IN LOGARITHMS)



- Recovered eigenfunction should be identical at every date!
- Constant interest rates (2011–2014?) consistent with a constant eigenfunction

  $\Longrightarrow$ evidence of misspecification

TAKEAWAYS

Two contributions

- theory: new conditions for uniqueness of the Hansen–Scheinkman decomposition
- numerical/empirical: extraction of measure P^ϕ with the help of B-spline fitting

What to do with these results?

- Combine with other (time series or even survey) data and/or more theory
- Acknowledge $P^\phi \neq P$ and use P^ϕ and ϕ for pricing of relevant cash flows
- How much additional information does P^ϕ bring relative to Q?
    - ϕ does absorb some risk adjustments but how much?