Online Appendix

Stability of Equilibrium Asset Pricing Models: A Necessary and Sufficient Condition

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1. Introduction

This online appendix, supplementary to the paper *Stability of Equilibrium Asset Pricing Models: A Necessary and Sufficient Condition*, outlines how the discrete time results on asset pricing in that paper can be extended to the continuous time setting.

The results shown below are only preliminary. For example, we minimize technical issues by focusing on a finite state space. A genuinely interesting study would go beyond this case, in order to allow state processes driven by diffusions. This seems quite possible, since many of the results we draw on hold for very general settings.

To summarize the results in this note, recall that the original paper shows that, in the discrete time case with stationary dividends, existence and uniqueness of equilibrium asset prices depends on the sign of the stability exponent

$$\mathcal{L}_\Phi := \lim_{n \to \infty} \frac{1}{n} \ln \pi_n,$$

where π_n is the current price of a zero coupon bond maturing at date n. Below, we introduce a continuous time counterpart of $\mathcal{L}_\Phi$, denoted by ℓ, and show that, in standard environments, $\ell < 0$ is a necessary and sufficient condition for existence and uniqueness of equilibrium asset prices. Hence, the continuous time results closely parallel the discrete time results.

There is also an added advantage of working in continuous time: a local description of the evolution of values encoded in the infinitesimal generator of the pricing semigroup. We connect our main results to that local description in Section 2.3. However, we treat no applications, leaving this as an important avenue for future research.

Our notation in this online appendix loosely follows Qin and Linetsky (2017).

2. The Continuous Time Case

Let E be a finite set and let $\mathbb{R}^E$ denote the set of all functions from E to $\mathbb{R}$. For $g \in \mathbb{R}^E$, we write

- $g \geqslant 0$ if $g(x) \geqslant 0$ for all $x \in E$,
- $g > 0$ if $g \geqslant 0$ and $g(x) > 0$ for some $x \in E$, and
- $g \gg 0$ if $g(x) > 0$ for all $x \in E$.

Let $(X_t)_{t \geqslant 0}$ be a stationary right-continuous Markov process taking values in E, defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and adapted to filtration $(\mathscr{F}_t)$. Let $(S_t)_{t \geqslant 0}$ be an everywhere positive $\mathscr{F}_t$-adapted process, to be interpreted as a pricing kernel process. When the current state is x, the current price of a time t payoff G_t is given by $\mathbb{E}_x S_t G_t$.

Given $\tau \geqslant 0$, let P_τ be the time τ spot price of a claim to a cash flow $(G_t)_{t \geqslant \tau}$ with no termination date. This price process $(P_t)_{t \geqslant 0}$ satisfies the intertemporal consistency condition

$$P_0 = \int_0^t \mathbb{E}_x S_u G_u \, \mathrm{d}u + \mathbb{E}_x S_t P_t \quad \text{for all } t \geqslant 0. \tag{1}$$

(See, e.g., Garman (1985).) From now on, we focus on stationary cash flows (G_t) driven by the state process, in the sense that there exists a $g \in \mathbb{R}^E$ with $G_t = g(X_t)$ for all $t \geqslant 0$. Let $\mathscr{P}_t$ be defined at all t by

$$(\mathscr{P}_t g)(x) := \mathbb{E}_x S_t g(X_t) \qquad (x \in E).$$

This represents the time zero price of the time t payoff $G_t = g(X_t)$, given current state x. Since S_t is everywhere positive, $g \mapsto \mathscr{P}_t g$ is a positive linear operator on $\mathbb{R}^E$ for all t. By the law of iterated values, $(\mathscr{P}_t)$ is an operator semigroup.[1]

After inserting the conjecture $P_t = p(X_t)$ for some fixed $p \in \mathbb{R}^E$, condition (1) becomes

$$p(x) = \int_0^t (\mathscr{P}_u g)(x)\, \mathrm{d}u + (\mathscr{P}_t p)(x) \quad \text{for all } t \geqslant 0. \tag{2}$$

Given $g \in \mathbb{R}^E$ with $g \geqslant 0$, we say that $p \in \mathbb{R}^E$ *prices* g if p satisfies (2) for all $x \in E$.

We discuss now necessary and sufficient conditions under which finite prices exist. After stating the main result we connect it to restrictions on infinitesimal descriptions of that process.

2.1. **Main Stability Result.** Since (X_t) is right continuous, the semigroup $(\mathscr{P}_t)$ is strongly continuous. In fact, since E is finite, $(\mathscr{P}_t)$ is norm continuous. Let ℓ be the stability exponent defined by

$$\ell := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathscr{P}_t\|. \tag{3}$$

We have the following result, which shows how the main theorem from discrete time can be translated into the continuous time case.

Theorem 2.1. *If (X_t) is irreducible, then the following statements are equivalent:*

(a) $\ell < 0$
(b) For all $g > 0$, there exists a $p \gg 0$ such that p prices g.
(c) For some $g > 0$, there exists a $p \geqslant 0$ such that p prices g.

Notice the strong converse implication: if $\ell \geqslant 0$, then there does not exist a nontrivial stationary payoff stream that can be priced under $(\mathscr{P}_t)$.

The proof of Theorem 2.1 can be found in Section 3.

[1] This follows from a natural arbitrage-based restriction on the pricing kernel (S_t). See Garman (1985), Hansen and Scheinkman (2009), or Section 7 of Anderson et al. (2003).

2.2. **Risk Neutral Pricing.** At this stage we give only a very simple example: risk neutral pricing. In particular, we assume that $S_t = \exp(-\rho t)$ for some constant ρ. It follows that, for any $g \in \mathbb{R}^E$,

$$(\mathscr{P}_t g)(x) = \exp(-\rho t)\mathbb{E}_x g(X_t) = \exp(-\rho t)(\mathcal{Q}_t g)(x),$$

where $(\mathcal{Q}_t)$ is the semigroup of transitions for the Markov process (X_t). Since each $\mathcal{Q}_t$ is a Markov operator, it satisfies $\|\mathcal{Q}_t\| = 1$ for all t, and hence the exponent ℓ satisfies

$$\ell = \lim_{t\to\infty} \frac{1}{t} \ln \|\exp(-\rho t)\mathcal{Q}_t\| = -\rho. \tag{4}$$

Hence, $\ell < 0$ if and only if $\rho > 0$. In other words, in the risk neutral case, all prices are finite if and only if the discount rate is positive.

2.3. **An Infinitesimal Approach.** In continuous time we can also apply an infinitesimal approach that has no parallel in discrete time. To describe it, we let $\mathscr{A}$ be the generator of $(\mathscr{P}_t)$. This operator is bounded, defined everywhere on $\mathbb{R}^E$ and satisfies $\mathscr{P}_t = \exp(t\mathscr{A})$ for all $t \geqslant 0$, due to norm continuity of $(\mathscr{P}_t)$. The next lemma is a version of a result presented in Garman (1985). Equation (5) is a generalized Poisson equation.

Lemma 2.2. . *If $g \in \mathbb{R}^E$ with $g \geqslant 0$, then p prices g if and only if*

$$-\mathscr{A} p = g. \tag{5}$$

Proof. Fix $g \in \mathbb{R}^E$ with $g \geqslant 0$. Suppose first that p prices g. By (2) we have

$$-\frac{1}{t}(\mathscr{P}_t p - p) = \frac{1}{t}\int_0^t \mathscr{P}_u g \, \mathrm{d}u \quad \text{for all } t > 0.$$

Taking $t \downarrow 0$ and recalling that $(1/t)(\mathscr{P}_t - I) \to \mathscr{A}$ everywhere on $\mathbb{R}^E$, we see that p solves (5). Conversely, if p solves (5), then integration gives $-\int_0^t \mathscr{P}_u \mathscr{A} p \, \mathrm{d}u = \int_0^t \mathscr{P}_u g \, \mathrm{d}u$ at all t. Standard results on strongly continuous semigroups (see, e.g., Engel and Nagel (2000)) give $\int_0^t \mathscr{P}_u \mathscr{A} p \, \mathrm{d}u = \mathscr{P}_t p - p$, so p prices g. □

Let $\sigma(\mathscr{A})$ be the spectrum of $\mathscr{A}$ and let $s(\mathscr{A}) = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathscr{A})\}$. The value $s(\mathscr{A})$ is called the *spectral bound* of $\mathscr{A}$.

Lemma 2.3. *Under the stated assumptions we have $s(\mathscr{A}) = \ell$.*

Proof. This follows from finiteness of E, which implies norm continuity of $(\mathscr{P}_t)$ and hence $s(\mathscr{A}) = \ell$ by Corollary IV.2.4 of Engel and Nagel (2000). $\square$

By combining Lemma 2.3 and Theorem 2.1, we see that the pricing semigroup uniquely prices all dividend processes if and only if $s(\mathscr{A}) < 0$.

Remark 2.1. Our guess is that Lemma 2.3 will provide the most useful approach for analyzing existence and uniqueness of asset prices in most continuous time quantitative applications. The reason is that, for pricing kernels driven by diffusions, the infinitesimal description is typically more tractable than the semigroup.

3. Remaining Proofs

All proofs are now completed.

Lemma 3.1. *If (X_t) is irreducible, then $(\mathscr{P}_t)$ is eventually strongly positive, in the sense that, for each $g \in \mathbb{R}^E$ with $g > 0$, we have $\mathscr{P}_t g \gg 0$ for all $t > 0$.*

Proof. Let (X_t) be irreducible and fix $g \in \mathbb{R}^E$ with $g > 0$. Pick any $x \in E$. By irreducibility, for each $y \in E$, we have $\mathbb{P}\{X_t = y\} > 0$ for all $t > 0$. Hence, fixing $t > 0$, we have $m(y) := \mathbb{E}_x S_t \mathbb{1}\{X_t = y\} > 0$ for all $y > 0$. As $g \geqslant 0$ and $g(y) > 0$ for some $y \in E$, we see that $(\mathscr{P}_t g)(x) = \sum_y g(y) m(y) > 0$. $\square$

Lemma 3.2. *If (X_t) is irreducible and, for some $g > 0$, there exists a $p \geqslant 0$ such that p prices g, then $s(\mathscr{A}) < 0$.*

Proof. Fix $g > 0$ and $p \geqslant 0$ such that p prices g. Since $g > 0$ and $(\mathscr{P}_t)$ is a strongly positive semigroup, by the definition in (2) and Lemma 3.1, we see that $p \gg 0$. By Theorem 1.1 of Daners et al. (2016), combined with strong positivity of $(\mathscr{P}_t)$, we have the following version of the Perron–Frobenius theorem: there exists an $e \in \mathbb{R}^E$ with $e \gg 0$ and $\mathscr{A}' e = s(A) e$ where $\mathscr{A}'$ is the adjoint of $\mathscr{A}$. From the Poisson equation (5) and the definition of the adjoint, $\langle \mathscr{A}' e, p \rangle = \langle e, \mathscr{A} p \rangle = -\langle e, g \rangle$. From $p, g > 0$ and $e \gg 0$, we have $s(\mathscr{A}) = -\langle e, g \rangle / \langle e, p \rangle < 0$. $\square$

Lemma 3.3. *If (X_t) is irreducible and $s(\mathscr{A}) < 0$, then, for all $g > 0$, there exists a $p \gg 0$ such that p prices g.*

Proof. Fix $g > 0$. Since E is finite, $(\mathscr{P}_t)$ is norm continuous. Hence, by Theorem V.1.10 of Engel and Nagel (2000) and $s(\mathscr{A}) < 0$, the semigroup $(\mathscr{P}_t)$ is uniformly exponentially stable. By the Datko–Pazy theorem (e.g., Engel and Nagel (2000), Theorem V.1.8), this implies that $p := \int_0^\infty \mathscr{P}_u g \, \mathrm{d}u$ is everywhere finite on E. Since (X_t) is irreducible, $(\mathscr{P}_t)$ is strongly positive and hence, by $g > 0$, we have $p \gg 0$. Thus, to complete the proof, we need only show that p prices g.

Because $0 > s(\mathscr{A})$, 0 is in the resolvent set of $\mathscr{A}$, and hence, by Theorem II.1.10 of Engel and Nagel (2000), we have $\int_0^\infty \mathscr{P}_u g \, \mathrm{d}u = -\mathscr{A}^{-1} g$. Hence $-\mathscr{A} p = g$, and p prices g by Lemma 2.2. □

Proof of Theorem 2.1. That (a) $\implies$ (b) follows from Lemmas 2.3 and 3.3. That (b) $\implies$ (c) is obvious. That (c) $\implies$ (a) follows from Lemmas 2.3 and 3.2. □

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