Lars Peter Hansen (1952–)

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Lars Peter Hansen was born on October 26, 1952 in Urbana, Illinois. When he was 15, his family moved to Utah, where his father became provost of Utah State University. While, according to his own words, he was an '*undistinguished and not-easy-to-deal-with*' high school student (Hansen 2013a), he started appreciating academia substantially more during his undergraduate years at Utah State. After graduating with a degree in mathematics and political sciences, strengthened by a range of courses in economics, he enrolled in the Ph.D. program at the University of Minnesota in 1974.

During the 1970s, the University of Minnesota was at the forefront in developing a new dynamic macroeconomics built on rational expectations. This transformation required new methodological approaches in macroeconometrics that would allow comparisons between model predictions and data, and Lars's research was immediately inspired by these advances. He started working with two young professors who had recently joined the economics department, Christopher Sims and Thomas Sargent. The synergy between Sims and Sargent in developing the links between modern macroeconomics and time series econometrics shaped research and education at Minnesota and had a tremendous impact on Lars. Sims became Lars's Ph.D. thesis advisor, while Sargent hired Lars as his research assistant, a move that signified the beginning of a lifelong collaboration.

Lars's Ph.D. thesis, titled '*Econometric Modeling Strategies for Exhaustible Resource Markets with Applications to Nonferrous Metals*', reflects the mentorship of both Sims and Sargent, and, in a certain way, indicates the directions of much of Lars's subsequent research. In the thesis, Lars investigates dynamic forward-looking models with restrictions implied by rational expectations,

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which he soon elaborated on in a series of papers with Sargent. The established moment restrictions are tested using financial markets data from London Metal Exchange, a precursor to Lars's quantitative work in asset pricing. The thesis is particularly concerned with the choice of proper model specification, later extensively reflected in his research on model uncertainty. Finally, the last two chapters establish formal properties of a class of generalized method of moments (GMM) estimators, to a great degree inspired by the discussion of these estimators in Sims' econometrics class (Hansen 1978, p.89). These chapters subsequently gave rise to the 1982 Econometrica paper on GMM.

Lars graduated from University of Minnesota in 1978, and joined the Graduate School of Industrial Administration at Carnegie Mellon University as assistant professor, followed by a quick promotion to associate professor. At Carnegie Mellon, he started fruitful collaborations with Kenneth Singleton, Robert Hodrick, and Scott Richard that advanced finance theory and extended the tools of rational expectations macroeconometrics to the analysis of asset pricing models.

In 1981, Lars joined the Department of Economics at the University of Chicago, where he remains. He was promoted to professor in 1984, served as the department chair, and later also received professorships at the Department of Statistics and at the Booth School of Business. He is currently the David Rockefeller Distinguished Service Professor at the University of Chicago.

Lars became instrumental in the econometrics group at the department, as well as a key uniting figure between the department and the finance program at the business school. He co-founded and continuously mentors the joint Ph.D. program in financial economics, and chaired the dissertation committee of more than sixty doctoral students. He was appointed as the inaugural director of the Becker Friedman Institute for Economics, directs the Macro Finance Research Program at Chicago, and, together with Andrew Lo from MIT, leads the Macro Financial Modeling Project. He served as president of the Econometric Society, vice-president of the American Economic Association, and co-editor of Econometrica.

Lars's multifaceted work was acknowledged through numerous awards, culminating in the 2013 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel, which he shared with Eugene Fama and Robert Shiller '*for their empirical analysis of asset prices*'. A personal account of the academic path and research agenda underlying the award can be found in Hansen (2013a) and Hansen (2014), and summaries connecting the seminal research contributions of the three recipients in Economic Sciences Prize Committee (2013a) and Campbell (2014). Lars also received the 2006 Nemmers Prize in Economics, the 2008 CME Group-MSRI Prize in Innovative Quantitative Applications, and the 2010 BBVA Foundation Frontiers of Knowledge Award. He is a fellow of the Econometric Society, American Academy of Arts and Sciences, National Academy of Sciences and the American Finance Association. Among his honorary degrees, he is particularly fond of the honorary doctorate from his alma mater, Utah State University.

Lars met his future wife, Grace Tsiang, when she was a graduate student at the University of Chicago, and they married in 1984. Grace had an immense impact on shaping the outstanding undergraduate program in economics at the university, and plays an equally important role in Lars's life, often providing first-hand feedback on his research and lectures. Their son Peter obtained a bachelor and a master degree from the University of Chicago and graduated with a Ph.D. in finance from MIT.

Just like his Ph.D. thesis, Lars's contributions to research in economics and finance are broad and intertwined. Lars studies how economic agents cope with risk and uncertainty, what are the implications of these decision problems for the economy, and what tools and approaches econometricians can use to analyze such economies. This essay traces out the evolution of this research agenda, in which the decision makers *inside* the model and the econometrician *outside* the model have been facing progressively more challenging environments that they had to navigate.

Section 1 of this essay starts with Lars's seminal work on Generalized Method of Moments estimation. The GMM methodology laid foundations for applied econometric work on partially identified models across the fields in economics, and is prominently cited by the Economic Sciences Prize Committee (2013a) for its role in advancing theoretical and empirical understanding of asset markets. Sections 2 and 3 discuss Lars's research in asset pricing, exploring the connections to macroeconomics and formal econometric evaluation of asset pricing models. His original contributions constitute some of the initial building blocks of a field that is now called macro-finance (Cochrane 2017), later work brought methodological contributions for the study of the term structure of risk in dynamic models. Section 4 returns to Lars's early collaboration with Thomas Sargent on rational expectations econometrics, to motivate departures from rational expectations toward model misspecification concerns. Section 5 deals with model misspecification on the side of the econometrician, section 6 describes these doubts on the side of agents inside the model, and section 7 discusses more policy-oriented work, in particular on uncertainty in the context of financial complexity and regulation, and on economic implications of climate change. Section 8 concludes.

1 Generalized method of moments

The development of dynamic equilibrium models in macroeconomics in the 1970s lead to increased interest in econometric methods that would implement testable restrictions implied by these models. Seminal work by Muth (1961), Lucas and Prescott (1971), Lucas (1972b) or Sargent (1973) on rational expectations models has shown how the solution to a fully specified equilibrium model takes the form of a stochastic process for the equilibrium path, with dynamics restricted by model equations that represent physical and institutional restrictions, optimizing behavior of economic agents, and equilibrium market forces. These *cross-equation restrictions* can be interpreted as a generalization of exclusion restrictions used in supply–demand analysis, and they determine the impact structural disturbances have on endogenous model variables.

Observable equilibrium dynamics are fully described by the likelihood function for the data. While likelihood-based estimation of model parameters in a correctly specified model achieves efficiency, there are appealing reasons for considering alternative methods. First, likelihood-based estimation requires a complete specification of the stochastic properties of the model, including aspects that are perhaps not essential to the question that the model is supposed to answer, or may not be as trusted by the researcher. Second, dealing with the likelihood function can often be computationally demanding. Instead, moment-based estimation allows researchers to focus their attention on implications of the model that they consider to be important, without having to spell out all details of the model, and often with considerable saving of computational costs. It allows the researcher to '*do something without having to do everything simultaneously*' (Clement and Hansen 2015).

The conceptual idea of moment-based estimation is straightforward. Consider a model that describes the joint dynamics of economic variables whose realizations at time t are summarized by the vector x_t, and features q unknown parameters of interest collected in a vector θ. Suppose we find r theoretical restrictions that observable variables in the model have to satisfy that can be written in vector form as

$$\mathbb{E}\left[f\left(x_t;\theta\right)\right]=0. \tag{1}$$

Here, f is a known function implied by the model, and $\mathbb{E}$ is the mathematical expectations operator under the data generating measure for x_t. These restrictions are called *moment conditions* because they represent various unconditional moments that the model variables x_t have to satisfy. The theoretical restrictions are implemented using time series data on x_t by approximating the theoretical expectations with sample averages over a sufficiently long period of time.

When we have the same number of restrictions as unknown parameters, $r=q$, then, under some regularity conditions, we can estimate the unknown parameter θ by finding a unique solution to the empirical counterpart of equation (1). But how should we proceed when $r>q$? In that case, all r *theoretical* restrictions still have to hold but that will typically not be the case in the finite data sample, for any parameter value θ. Moreover, how should we conduct *inference*, in other words, how should we deduce what are the distributional properties of estimated values of θ in order to be able to conclude whether the estimate is statistically close to a theorized value?

Lars's seminal paper on large sample properties of GMM estimators that provides answers to these questions was published by Econometrica in 1982 (Hansen 1982), with some details and proofs omitted from the original publication published 30 years later, in Hansen (2012c). The importance of the paper lies in expanding and connecting two crucial conceptual ideas which allow for a very broad range of applications, not only in the time series context described above but in cross-sectional applications as well.

The first is the idea of selection matrices, used in the analysis of instrumental variable (IV) estimators in Sargan (1958, 1959) and presented in Sims' econometrics class. Selection matrices serve as a device for pre-selecting linear combinations of the moment conditions, and then choosing them in an optimal way that minimizes the covariance of the estimator. The second contribution is the extension of results by Gordin (1969), Scott (1973) and others in asymptotic theory for stationary ergodic processes. Using these results, Lars was able to prove consistency and characterize in an elegant way distributional properties of the estimator for a range of environments with temporally dependent errors.

The versatility of the framework also allows to cast a number of other estimators, including least squares, instrumental variable estimators, or maximum likelihood, in the form of a GMM estimator. The GMM technique thus proved to be an extremely useful approach for a diverse set of problems, ranging from estimation of factor models in asset pricing (Jagannathan and Wang (1996), MacKinlay and Richardson (1991), and the textbook treatment in Cochrane (2005)), monetary policy rules (Clarida et al. (2000)), and discrete-choice models (Avery et al. (1983), Berry et al. (1995)), to treating panel data (Arellano and Bond (1991)). An extensive summary of the proliferation of the GMM methodology into econometric theory and applications in macroeconomics, finance, and other fields is provided in the contributions to Issue 4 of the 2002 volume of Journal of Business & Economic Statistics, as well as in the handbook chapter of Ogaki (1993).

At the same time, the formulation of the GMM estimator opened a range of conceptual aspects that Lars and others continued to develop, some of which are summarized in Hansen (2001, 2008). The GMM estimator can be thought of as the solution to the minimization problem

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} T g_T(\theta)' W g_T(\theta) \tag{2}$$

where Θ is the parameter space, T the sample size, $g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} f(x_t; \theta)$ is the sample counterpart of the moment in (1), and W is a suitably chosen weighting matrix, directly related to the choice of the selection matrix discussed above.

The asymptotically efficient choice of W under a correctly specified model corresponds to the inverse of the long-run covariance matrix of the moment conditions under the true parameter value. This covariance matrix must typically be estimated from finite samples of data, for example using an estimator based on spectral density at zero frequency as in Newey and West (1987), and the form of the moment conditions implies that this estimate will depend on the estimated parameter vector θ. This led Hansen et al. (1996) to investigate the small-sample properties of alternative GMM estimators that differ in the construction of the long-run covariance matrix, motivated with applications from the asset pricing literature. In addition, the whole Issue 3 of the 1996 Journal of Business & Economic Statistics is devoted to studying various aspects of small-sample properties of the GMM estimation method in a range of contexts.

Another important question in GMM estimation is the choice of moment conditions (Gallant and Tauchen 1996). In many environments, the set of available moment conditions is large and potentially infinite, for example due to a large set of instruments available as conditioning variables, their arbitrary transformations, as well as lagged variables in time series applications. Hansen (1985) and Hansen et al. (1988) show how to construct the greatest lower bound on the asymptotic covariance of such a class of estimators. Hansen and Singleton (1991, 1996) and Nagel and Singleton (2011) specialize this analysis to linear time series models and asset pricing applications.

The above choices of moment conditions and weighting matrices strive to achieve statistical efficiency under a correctly specified model. On the other hand, alternative specifications of the selection matrix and the resulting weighting matrix W may be motivated by other considerations, tightly related to Lars's work on econometric analysis of misspecified models that we elaborate on in Section 5. This flexibility allows to exploit GMM as a unifying framework for a range of approaches in time series econometrics, macroeconomics, and asset pricing.

For example, a popular technique in the macroeconomic literature is based on calibration of model parameters using a set of 'target' moments and subsequent evaluation of model performance using an alternative set of 'untargeted' ones (Kydland and Prescott 1996, Hansen and Heckman 1996, Evans et al. 2005). This approach emphasizes the econometrician's prior knowledge in judging which combinations of moment conditions are particularly revealing for the questions the model is designed to answer. The procedure can be formalized by choosing a selection matrix that subselects the desired target moments in (1). The GMM counterpart to the verification stage then consists of an overidentification test based on the remaining moments (Christiano and Eichenbaum 1992, Hansen 2008).

Choosing the weighting matrix to be the inverse of the long-run covariance matrix evaluated at the estimated parameter value rewards moments and parameter estimates that lead to large sampling variability. Such a criterion is driven by statistical efficiency but it may be undesirable when selected moments have a particular economic interpretation. In the asset pricing literature, these moments often come in the form of pricing errors generated by misspecified valuation models. Hansen et al. (1995) and Hansen and Jagannathan (1997) interpret such pricing errors as moments which they use to construct a measure of misspecification of the valuation model, and use a weighting matrix that normalizes the magnitude of payoffs, rather than variability of the pricing errors.

Hansen and Scheinkman (1995) showed how to utilize the GMM methodology in estimation of continuous-time Markov processes. These processes can be characterized via infinitesimal generators which specify their local properties. Dynamics over longer time periods are then constructing by integrating these generators up over time. The paper focuses on two types of moment conditions constructed from the generator, one implied by the stationary distribution of the data, and another implied by the conditional density of the evolution of the process in the forward and backward direction. A key input lies in imposing restrictions on the generator such that a version of the Central Limit Theorem applies to discretely sampled data, relating these results to Lars's earlier work on time-aggregation and anti-aliasing challenges when identifying continuous-time models using discrete-time observations (Hansen and Sargent 1983a,b). Equipped with these results, the moment conditions can then be used as an input into a GMM estimator. We return to applications of this framework in the context of estimation of misspecified models in Section 5.

The typical application of GMM utilizes the quadratic form on the right-hand side of (2) to construct an overidentification test based on Pearson (1900)'s chi-squared distributed statistic. Arellano et al. (2012) instead devise a technique that exploits this quadratic form to test for a specific type of *underidentification*. Imagine a curve of alternative values of θ in the parameter space for which the moment conditions (1) hold. We can then form a stacked vector of moments by repeating moment conditions (1) evaluated at a finite set of values of θ lying on the curve. An overidentification test on this stacked system that is not able to reject the null hypothesis that all conditions hold simultaneously then in fact provides evidence of underidentification.

These are just some examples of the wide-ranging applications of the GMM approach, which became truly ubiquitous in economics. The methodology also plays a prominent role in Lars's asset pricing research, which we explore next.

2 Testing asset pricing models

Joining the business school at Carnegie Mellon University in 1978 provided Lars with an important impetus to engage in work on asset pricing. He arrived at a department skewed toward younger faculty, and started productive research collaborations with Robert Hodrick, Scott Richard, and Kenneth Singleton, later also joined by Martin Eichenbaum. This gave rise to several key contributions in testing asset pricing restrictions imposed by competitive equilibrium models of financial markets, relying in essential ways on the GMM methodology.

Asset pricing applications are particularly suitable for estimation and testing using GMM because the return R^i_{t+1} between period t and $t+1$ on any traded asset i has to satisfy the restriction

$$\mathbb{E}_t\left[m_{t+1}R^i_{t+1}\right] = 1, \tag{3}$$

where $\mathbb{E}_t$ is the expectation conditional on time t information, and m_{t+1} is a strictly positive stochastic discount factor (SDF). Building on Ross (1978) and Harrison and Kreps (1979), Lars and Scott Richard showed how to use conditioning information to derive such restrictions in general dynamic economies (Hansen and Richard 1987). Applications that use this methodology for valuation of multiperiod cash flows in specific economies are then provided in Hansen (1987).

Restrictions in the form (3) are weak because they follow solely from linearity of prices and absence of arbitrage in financial markets. In order to provide stronger empirical content and make the restrictions testable, the researcher needs to impose a particular theory for the SDF. For example, in the presence of a marginal investor, m_{t+1} is identified with the marginal rate of substitution of this investor that encodes time preference and risk aversion, and $\mathbb{E}_t$ with the subjective belief of the investor. Equation (3) then represents an intertemporal optimality condition, stating that the investor has to be indifferent between one unit of consumption today and return R^i_{t+1} on this one unit next period, properly adjusted by the marginal rate of substitution.

The literature in macroeconomics and finance proceeded by specifying particular models of investors' marginal rate of substitution m_{t+1}, parameterized by a vector of structural parameters θ. We return to decision problems that yield such models of m_{t+1} in Sections 4 and 6. In order to construct tests of these theories, one can multiply (3) with a vector of instruments or conditioning variables z_t, and take unconditional expectations. Under rational expectations, the investors' subjective belief is identical to the data-generating probability measure, and we obtain the moment condition (1) with $f\left(x_{t+1};\theta\right) = \left(m_{t+1}\left(\theta\right)R_{t+1} - 1\right) \otimes z_t$, where R_{t+1} is the vector of all returns i considered for the test.

In Hansen and Hodrick (1980), Lars and Robert Hodrick used data from foreign exchange markets to test market efficiency under the assumption that the marginal rate of substitution is, in nominal terms, conditionally uncorrelated with returns on forward contracts (this would occur, for example, if investors were risk neutral and inflation conditionally uncorrelated with movements in exchange rates). While the empirical evidence they obtained was mixed, a key contribution is the empirical implementation of econometric results developed in Hansen (1979), which increased statistical efficiency in a situation when regressors are not strictly exogenous and future realizations of forecasting variables are correlated with forecast errors of current forecasts. In Hansen and Hodrick (1983), they extended the analysis to tests of models with linear risk premia.

Inspired by the work of Hall (1978) and Grossman and Shiller (1981) on the implications of consumption Euler equation restrictions, Hansen and Singleton (1982) and Hansen and Singleton (1983) set out to test Euler equation restrictions implied by risky and risk-free investment opportunities available to a U.S. investor, using GMM and maximum likelihood estimation, respectively. These two influential papers provided early convincing evidence on the quantitative challenges faced by consumption-based asset pricing models in rationalizing observed moments in asset returns. Hansen and Singleton (1982) won the 1984 Frisch Medal awarded biennially to the best applied paper published in Econometrica during the last five years.

The papers utilize the competitive representative agent environment of Lucas (1978) and Breeden (1979), and impose a model of the SDF derived from preferences with constant relative risk aversion (CRRA) over aggregate consumption C_t. This specification implies $m_{t+1} = \beta \left(C_{t+1}/C_t \right)^{-\gamma}$, where β is the time preference and γ the risk aversion of the representative investor. The results show that overidentifying restrictions based on conditional moments and multiple assets with heterogeneous riskiness lead to convincing rejections of the model. Rejections based on conditional moments imply that the model of the SDF does not reflect variation in conditional expected returns observed in the data. On the other hand, rejections based on multiple assets imply that the required compensation for risk implied by the SDF is too low to rationalize differences in average returns across these assets. To see this, notice that a simple manipulation of (3) applied to a risky return R^i_{t+1} and the (gross) risk-free rate R^f_t yields

$$E_t \left[R^i_{t+1} \right] - R^f_t = -R^f_t Cov_t \left(m_{t+1}, R^i_{t+1} \right). \tag{4}$$

Differences in expected returns in excess of the risk-free rate therefore have to be attributed to differences in the covariance of the SDF with those returns. The failure of the CRRA model comes from the fact that the observed consumption growth C_{t+1}/C_t is not sufficiently volatile and correlated with observed returns to generate a meaningfully large covariance in (4), unless the risk aversion parameter γ is assumed to be implausibly high.

These results tie together an important set of contemporaneous research that critiques early models of the SDF. LeRoy and Porter (1981), Shiller (1981), and Grossman and Shiller (1981) focus on the large volatility of stock prices and conclude that this volatility cannot be rationalized without a model that generates large variations in risky discount rates. Mehra and Prescott (1985) then show that a plausibly calibrated model with CRRA preferences cannot generate the large observed equity premium in the aggregate stock market, the so-called equity premium puzzle, confirming results from Hansen and Singleton (1982, 1983) for a specific model of the aggregate consumption process. Weil (1989) argued that an additional puzzle in the form of a too low empirically observed risk-free rate emerges from this class of models as well.

These rejections of early asset pricing models have led to a fruitful research agenda that has enriched the models in a variety of ways and significantly improved their empirical fit, by boosting covariances of the SDF with returns in (4) but also in terms of other features like return predictability (Campbell and Shiller 1988, Lettau and Ludvigson 2001, Cochrane 1992, 2008, 2011). This development focused on introducing a richer structure of risk, e.g., using highly persistent shocks to consumption growth (Bansal and Yaron 2004) or disaster risk (Rietz 1988, Barro 2006); specifying preferences that implied larger and time-varying compensation required by investors for holding risky assets, e.g., in models with recursive preferences (Epstein and Zin 1989, 1991, Weil 1989), habit formation (Abel 1990, Constantinides 1990, Campbell and Cochrane 1999), risk-sensitive preferences (Tallarini 2000), or ambiguity aversion (Cagetti et al. 2002, Epstein and Schneider 2008, Ju and Miao 2012); relaxing the assumption of complete markets by modeling imperfect risk sharing (Constantinides and Duffie 1996), private information (Kocherlakota 1996), or by adding sources of nontradable risk (Heaton and Lucas 1996, 2000), to name a few important directions.

While the above studies took a distinctly asset pricing oriented approach that investigates asset prices primarily in the context of endowment economies as in Lucas (1978) or Breeden (1979), Lars also worked on connections between asset prices and the macroeconomy. In Eichenbaum et al. (1988), they estimated a model of nonseparable preferences over consumption and leisure that also allowed for nonseparability over time in the form of habits and durability effects. The purpose was to analyse the joint behavior of interest rates, consumption, wages and hours worked. Eichenbaum and Hansen (1990) studied substitutability patterns between durable and nondurable consumption. Such nonseparabilities in preferences, for example, between non-durable and durable consumption (Heaton 1993, 1995, Yogo 2006), home production (Benhabib et al. 1991), leisure (Jagannathan and Wang 1996), or housing (Piazzesi et al. 2007), now constitute well-regarded preference features in quantitative macroeconomics and finance.

Constructing a GMM estimator from the Euler equation restrictions represented major progress in testing asset pricing models because it did not require specifying details of the stochastic process driving returns or the stochastic discount factor. Yet it did require taking a stand on the form of the SDF and finding its observable counterpart—for example, the CRRA model specified in the preceding section required measurement of aggregate consumption growth.

In Hansen and Jagannathan (1991), Lars and Ravi Jagannathan used moments of observable asset returns to construct an informative restriction on the second moment of the SDF that *any* SDF that correctly prices these returns must satisfy. In its simplest form, it can be derived from (4) by noting that $Cov_t\left(m_{t+1}, R_{t+1}^i\right) = \sigma_t\left(m_{t+1}\right)\sigma_t\left(R_t^i\right)\rho_t$ where $\sigma_t\left(\cdot\right)$ denotes conditional volatility, and ρ_t is the correlation between m_{t+1} and R_{t+1}^i. Since correlation cannot exceed one, we obtain the inequality

$$\frac{\sigma_t\left(m_{t+1}\right)}{E_t\left[m_{t+1}\right]} \geq \frac{\left|E_t\left[R^i_{t+1}\right]-R^f_t\right|}{\sigma_t\left(R^i_t\right)}. \tag{5}$$

This restriction, which became familiar as the Hansen–Jagannathan (HJ) bound, can be studied in its conditional and unconditional form (for an implementation with efficient use of conditioning information see Gallant et al. (1990)), sharpened by exploiting the covariance structure in measured returns, and modified to restrict attention to only positive SDFs or to incorporate various types of market imperfections like borrowing constraints. Cochrane and Hansen (1992) provide an excellent pedagogical treatment with a range of applications.

The strength of the HJ bound (5) lies in separating observable properties of returns on the right-hand side from properties of the SDF m_{t+1}. The bound has to hold for any SDF that prices returns on the right-hand side, regardless of their economic foundations. Since the conditional mean $E_t\left[m_{t+1}\right]$ is equal to the reciprocal of the risk-free rate, the bound effectively states that Sharpe ratios, i.e., ratios of average excess returns over the return volatility (Sharpe 1966), constitute a lower bound on the volatility of the SDF.

Because the bound is based on very few assumptions, it could potentially be weak. It turns out that contemporaneous models of SDFs with parameters deemed plausible missed the bound by a factor of ten or more, even for the most generic returns R^i_{t+1} on the right-hand side, like the return on the aggregate stock market. This led to a substantial rethinking of approaches to SDF modeling, as large classes of models with low volatility of the SDF could be discarded as implausible from the onset.

Conceptual ideas underlying the volatility bound have been refined and applied in broader contexts. Burnside (1994), Cecchetti et al. (1994), and Otrok et al. (2002) conducted econometric analyses of the bound, while Otrok et al. (2007) generalized it to incorporate serial correlation in returns. He and Modest (1995) and Luttmer (1996) incorporated financial frictions into the construction of the bound, and Alvarez and Jermann (2001) used it to study the role of endogenous solvency constraints. Tallarini (2000) and Gomme et al. (2011) explore the implications of the bound in a business cycle environment.

Finally, the literature expanded beyond studying model implications for second-moment bounds. For example, Stutzer (1995), Bansal and Lehmann (1997), Backus et al. (2011) or Backus et al. (2014) propose to use entropy as a measure of dispersion of the SDF, while Almeida and Garcia (2017) explore the whole Cressie and Read (1984) family of discrepancy measures. Similar bounds were also constructed to constrain the magnitude of unobservable components of the SDF that are required for correct pricing of a desired vector of assets (Ghosh et al. 2017, Hansen 2014, Borovička et al. 2016), a topic that we revisit in the discussion of model misspecification in Section 5. The literature originated by Hansen and Jagannathan (1991) has thus evolved into a productive exploration of asset pricing implications of broad classes of stochastic discount factors.

3 Term structure of risk

Much of the analysis of asset pricing models described in Section 2 was focused on testing one-period relationships between returns and stochastic discount factors. Yet as asset pricing models evolved to be more sophisticated and able to rationalize at least static relationships between marginal rates of substitution and cross-sectional patterns in returns, attention started shifting toward their dynamic implications. Lars, in particular working with his colleague José Scheinkman and others, sought inspiration in tools from dynamic macroeconomic analysis to gain insights into implications of alternative models for the term structure of asset returns, in order to unite quantitative work on macroeconomic dynamics and asset pricing.

This led to development of novel methods specifically designed to account for nonlinearities and permanent growth components prominently featured in the asset pricing literature. Intellectual origins of these ideas can be traced back to the work of Fisher (1930) and Hicks (1939) on horizon-dependent effects of risk and discounting. An important empirical motivation also emerged from the more recent empirical literature on the term structure of returns on so-called dividend strips (van Binsbergen et al. 2011, 2013, van Binsbergen and Koijen 2017).

Lars's Fisher–Schultz Lecture (Hansen 2012b) provides a comprehensive overview of this research agenda about *dynamic valuation decomposition*. The idea is to study the contribution of non-diversifiable structural shocks to the pricing of risky cash flows with alternative maturities. A central result in this work is the multiplicative nonlinear decomposition of SDFs and cash flows into their permanent and transitory components derived in Hansen and Scheinkman (2009). Further methodological details are elaborated on in the handbook chapters of Hansen (2013b) and Borovička and Hansen (2016).

Perhaps the first traces of these methodological innovations appear in Hansen et al. (2005), which investigated time series properties of dividend growth using linear vector-autoregressions. The paper is motivated by the hypothesis, previously investigated by McGrattan and Prescott (2000) and Hall (2001), that unmeasured intangible capital is an important driving force of stock market valuations. If that is the case, systematic differences in book-to-market ratios in the cross-section of firms could be attributed to differences in the shares of measured tangible and unmeasured intangible capital stock.

To investigate this hypothesis in more depth, the authors studied the heterogeneity in risk exposures of dividend growth across portfolios of firms sorted on their book-to-market ratios. Similar to Bansal et al. (2005), they document that dividend growth of firms with high book-to-market ratios, so-called value firms, is substantially riskier compared to their low book-to-market counterparts, called growth firms. Hansen et al. (2005) pushed this story further by studying long-run properties of these cash flows using linear methods based on the Blanchard and Quah (1989) decomposition of the cash flows into permanent and transitory components. This further motivated their study of models that explicitly take into account heterogeneity in riskiness of tangible and intangible capital.

Since cash flow risk should be reflected in valuations, these results also have implications for cross-sectional patterns in risk premia. A large empirical literature initiated by Fama and French

(1992) has documented the so-called value premium, the observation that value firms earn higher average returns compared to growth firms, and that this return difference cannot be attributed to exposure to aggregate market risk.

The challenge for asset pricing theory is to attribute these differences in average returns, per equation (4), to covariances of returns on these alternative portfolios with the stochastic discount factor. As discussed in Section 2, previous work had uncovered the failure of simple consumption-based models with CRRA preferences over consumption to rationalize the magnitude of risk premia on risky assets, caused by insufficiently large covariances between contemporaneous consumption growth and asset returns.

This led Hansen et al. (2008) to investigate a class of non-separable recursive preferences introduced in Kreps and Porteus (1978) and Epstein and Zin (1989, 1991) in a model of long-run risk developed by Bansal and Yaron (2004). Due to its non-separable structure, this model of preferences breaks the tight link between the SDF and contemporaneous consumption growth encoded in CRRA preferences. Instead, investors' marginal rate of substitution m_{t+1} depends both on current consumption growth as well as on news about future consumption growth; in typical applications the latter channel quantitatively dominates. Risk premia are then largely driven by the covariance of news about future consumption and dividend growth, and the key empirical challenge shifts to identification of sufficiently strong predictability of future consumption and cash flows.

The work in Hansen et al. (2005) and Hansen et al. (2008) largely relies on applications of loglinear methods. The handbook chapter by Hansen et al. (2007) provides an excellent comprehensive summary of these methods in the context of recursive references. These loglinear methods were, however, insufficient for studying asset pricing implications in more sophisticated nonlinear models

As a major step forward in the analysis of term structure of risk in nonlinear models, Hansen and Scheinkman (2009) construct a multiplicative decomposition of cash flow and SDF processes that isolates a permanent component of the process with a long-horizon pricing impact. The method is based on solving a Perron–Frobenius eigenproblem and can be viewed as a nonlinear counterpart to the additive linear decomposition of Blanchard and Quah (1989). Specifically, they have shown how, in Markov models with a stationary state vector x_t, to decompose the one-period stochastic discount-factor as

$$m_{t+1} = \lambda \frac{e\left(x_t\right)}{e\left(x_{t+1}\right)} \widehat{m}_{t+1} \tag{6}$$

where λ is a constant that governs the deterministic drift of the SDF and corresponds to the principal eigenvalue of a particular valuation operator, $e\left(x_t\right)$ is a stationary component corresponding to the principal eigenfunction, and $\widehat{m}_{t+1}$ is a multiplicative increment to a martingale. The stationary component affects pricing over finite horizons, the yield on long-term bonds is determined by λ, and the risk premium on risky long-run cash flows is driven by the covariance of the martingale constructed by accumulating increments $\widehat{m}_{t+1}$ with a corresponding martingale extracted from the cash flow dynamics.

These results sharpened insights about the roles of permanent and transitory shocks in modern asset pricing models, including their quantitative importance. For example, using results from Kazemi (1992), the transitory component $\lambda e(x_t)/e(x_{t+1})$ is identified from the reciprocal of the holding-period return on long-run bonds. Alvarez and Jermann (2005) use these holding-period returns to bound the dispersion of the permanent component of the SDF without imposing a specific model of preferences. The handbook chapter of Hansen (2013b) provides a detailed summary of such implications. Hansen and Scheinkman (2017) also discuss a variety of other applications of the decomposition (6) in asset pricing, including the crucial role that the principal eigenvalue λ obtained from a continuation value recursion plays in determining existence of continuation utilities in the recursive preference model of Epstein and Zin (1989), analyzed in Hansen and Scheinkman (2012b).

How the Hansen and Scheinkman (2009) decomposition isolates the role of permanent shocks in asset pricing models also helped illuminate the so-called *recovery* result. Asset prices are determined by the composition of investors' preferences encoded in the SDF and their beliefs, through restrictions imposed by Euler equations (3). Economists have traditionally used combinations of time-series and cross-sectional data on asset prices and macroeconomic quantities, together with theoretical models that imposed restrictions on investors' beliefs and preferences, to uncover unknown model parameters. The work of Hansen and Singleton (1982, 1983) on testing specific models of stochastic discount factors described in Section 2 is an important contribution to this research program.

As a notable departure from this approach, Ross (2015) showed that if the SDF takes the form $m_{t+1} = \beta u'(x_{t+1})/u'(x_t)$ for some unknown time preference parameter β and marginal utility function $u'(x)$, one can at least theoretically deduce both the SDF and investors' beliefs from a sufficiently rich cross-section of asset prices, without using time-series data or imposing more structure on the function $u'(x)$. Borovička et al. (2016) compare this result with decomposition (6) and argue that imposing such a form of the SDF is equivalent to assuming that the martingale component $\widehat{m}_{t+1}$ is degenerate, i.e., identical to one.

Beliefs recovered from the cross-section of asset prices using this method are then distorted by the neglected martingale component $\widehat{m}_{t+1}$, which remains unidentified. The magnitude of the distortion must therefore be assessed using other information, either from parameterized models of preferences or from time-series data, for example, from average asset returns. Structural asset pricing models as well as evidence from preference-free bounds in Alvarez and Jermann (2005) and the follow-up literature imply that the martingale component plays a prominent quantitative role in the dispersion of the SDF, required to jointly reconcile large risk premia and low variability of short-term interest rates. This indicates a quantitatively large discrepancy between the recovered and true belief of the marginal investor.

Continuing the line of work on term structure of risk premia, Lars and his coauthors also developed a methodology that refined pricing implications of alternative economic shocks in asset pricing models. Borovička et al. (2011) and Hansen and Scheinkman (2012a) characterized so-called *shock elasticities*, functions that represent the sensitivity of cash flows and associated expected returns on these cash flows to contemporaneous shocks across alternative cash flow maturities. These shock elasticities describe horizon-dependence of the impact that a shock has on cash flows and asset prices and are specifically designed to account for nonlinearities embedded in asset pricing models. The long-horizon limits of these shock elasticities are closely related to properties of the martingale component of the SDF isolated in (6).

The methodology is again inspired by tools from macroeconomic analysis that originated in work of Slutsky (1927), Yule (1927), and Frisch (1933) on impulse and propagation problems in time series analysis. Impulse response functions are used extensively to study dynamic implications of macroeconomic models (Sims 1980), and the shock elasticities provide a unified treatment of the dynamics of macroeconomic quantities and the term structure of risk. Borovička and Hansen (2014a) apply them to study asset prices in production economies, revisiting questions about heterogeneous riskiness of tangible and intangible capital. Borovička et al. (2014) develop a formal connection between shock elasticities and nonlinear impulse response functions constructed in Koop et al. (1996) and Gourieroux and Jasiak (2005). The handbook chapter of Borovička and Hansen (2016) summarizes this work and elaborates on computational tools and further applications in discrete- and continuous-time frameworks.

The macro-finance literature has recently seen a proliferation of new models that deploy distinct mechanisms to rationalize similar sets of established asset pricing facts. The tools of dynamic valuation decomposition are particularly promising as a way of distinguishing between these alternative theories. As first steps in this direction, Zviadadze (2017, 2020) extends the shock elasticity methodology and uses it to test workhorse asset pricing models by comparing implications of identified structural shocks for the term structure of risk premia.

4 Rational expectations econometrics

During his graduate studies, Lars joined Thomas Sargent in developing methodological insights into solving and estimating dynamic macroeconomic models populated by decision makers endowed with rational expectations. Following earlier work on rational expectations econometrics (Lucas (1972a), Sims (1972), Sargent (1977, 1978b,c), Sargent and Sims (1977), Kennan (1979), Lucas and Sargent (1978)), they focused on the correlation structure in observed time series implied by cross-equation restrictions obtained from the solution of such models. These restrictions again emerge in the form of moment conditions (1) that tie together decision makers' choices and their forecasts of the evolution of relevant economic variables as equilibrium outcomes.

To illustrate the class of decision problems studied in this work, and to motivate later discussion, let us consider a single-agent linear quadratic optimal control problem from Anderson et al. (1996). In this problem, the decision maker faces a state vector x_t and has available a vector of controls u_t. The state vector encompasses variables that are exogenous to the decision maker, as well as those endogenously affected by the choice of u_t. Controls are chosen so as to maximize the present discounted value of utility flow in the form of a quadratic function $v(x_t, u_t)$, with time preference parameter β.

Following the insight that '*one Hamilton–Jacobi–Bellman equation is worth a thousand words.*' (Hansen et al. 2006, p.48), the infinite-horizon discrete-time problem can be written in recursive form as

$$V\left(x_t\right) = \max_{u_t} v\left(x_t, u_t\right) + \beta \mathbb{E}_t\left[V\left(x_{t+1}\right)\right], \tag{7}$$

where $V\left(x_t\right)$ is the value function and $\mathbb{E}_t$ is the conditional expectations operator, subject to the regulated law of motion

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}, \tag{8}$$

where $w_{t+1} \sim N\left(0, I\right)$ is a vector of Gaussian shocks, and A, B, C are conforming coefficient matrices. Under particular restrictions on the coefficients of the problem, a unique stable solution exists, characterized by a linear optimal decision rule $u_t = Fx_t$, and a quadratic value function $V\left(x_t\right)$. These decision rules and their aggregate implications form the cross-equation restrictions that restrict the joint dynamics of observable variables and can be exploited as moment conditions (1) for the GMM estimator.

In this rational expectations environment, the agents inside the model and the econometrician outside the model are endowed with substantial amount of knowledge. Rational expectations dictate that the beliefs of economic agents and the econometrician are aligned with the data-generating measure, determined in equilibrium as an endogenous outcome. The goal of the econometrician is to estimate unknown parameters of the model but the structure of the model is doubted neither by the agents nor by the econometrician. We revisit the structure of this problem in subsequent sections to show how it can be extended to account for model misspecification concerns on the side of the econometrician and the decision makers.

Hansen and Sargent (1980, 1982) addressed environments in which the econometrician does not observe some of the variables that are relevant for the decision maker's forecasting problem. Under the econometrician's information, this may lead to seemingly puzzling situations when endogenous choices statistically predict exogenous 'forcing' variables in the sense of Granger (1969). Estimating structural relationships between endogenous variables using least squares then produces biased results due to the failure of strict exogeneity of regressors that may depend on leads and lags of the error terms. This work provides ways how to apply GMM advantageously in a way that allows accounting for temporal dependence in errors, to overcome this lack of strict exogeneity. Hansen and Sargent (1981a,b) then also developed prediction formulas and efficient computational methods for this class of models. Hansen et al. (1985) apply these methods in the context of strategic duopolies in industry dynamics subject to resource depletion.

An extensive and pedagogically oriented collection of essays focused on analysis of linear rational expectations models is assembled in the *Rational Expectations Econometrics* volume (Hansen and Sargent 1991a), with contributions from William Roberds, John Heaton and Albert Marcet. These include discussion of least squares prediction theory, construction and estimation of structural models, interpretation of vector autoregressions in the context of rational expectations restrictions, forecasting formulas and solution algorithms for continuous-time versions of these models, and implications of temporal aggregation.

Particularly important are the chapters by Hansen and Sargent (1991b) on fundamental solutions to vector-autoregressions, and by Hansen et al. (1991) on the time-series implications of present value budget constraints. The former shows that vector-autoregressions estimated without imposing economic restrictions using data drawn from an equilibrium process may not uncover true economic shocks faced by economic agents, an issue known as the invertibility problem (Fernández-Villaverde et al. 2007). The latter chapter demonstrates that present value relations in models of household saving (Hall 1978), government surpluses (Barro 1979) or dividends (Campbell and Shiller 1987, 1988) cannot be identified from cash flow data alone without additional restrictions, because these data again do not reveal agents' information sets (Cochrane 2001).

Specific computational tools can be employed when the equilibrium dynamics are obtained from the solution of a single-agent optimal control problem. Anderson et al. (1996) summarize the analysis of such optimal control problems in linear-quadratic environments. These computational algorithms are based on solving recursive models in the form of matrix Riccati and Sylvester equations for their fixed points. Together with key results in Blanchard and Kahn (1980), Sims (2002) and elsewhere, these methods are designed to combine initial conditions for state variables with stability restrictions imposed on forward-looking variables to obtain a unique stable solution that determines the equilibrium path. Fast algorithms with semianalytical solutions make estimation of these models particularly tractable, and they are nowadays in the toolbox of every modern macroeconomist, see, for example, the widely used textbook by Ljungqvist and Sargent (2018).

5 Econometric analysis of misspecified models

The discussion in the preceding sections has largely put aside an important aspect of Lars's work focused on model uncertainty. Neither the econometrician constructing moment conditions (1) for the GMM estimator, nor agents who solved decision problem (7)–(8), and whose beliefs about future implied asset pricing restrictions (3), doubted the model of the economy. Dealing with such doubts is an inherent part of Lars's research program. Even though he has viewed cross-equations restrictions emerging from rational expectations models as a productive way of moving research in macroeconomics and asset pricing forward, he has always stressed the importance of understanding these models as an informative but simplified and imperfect description of reality. Rather than discarding imperfect models, he strived to adapt modeling methods in order to obtain appropriate interpretations of their implications.

Concerns about potential model misspecification in econometric analysis were clearly reflected already in Lars's Ph.D. thesis. He refers to Leamer (1978)'s concept of specification searches when constructing models that are appropriate simplifications of reality (Hansen 1978, p.22):

> [I]t is difficult to precisely specify prior information and not feasible to completely delineate the mapping from priors to posteriors. Practically speaking, it is necessary to

> reduce one's attention to an interesting class of approximate prior distributions. The rational expectations methodology provides a means of appealing to dynamic economic theorizing to obtain approximate restrictions on the vector autoregressive representations of economic time series.

In his Nobel Prize Lecture (Hansen 2014), Lars distinguishes between two types of model misspecification that he calls outside and inside uncertainty. *Outside uncertainty* refers to econometrician's or policy-maker's concerns that the estimated model of the economy is misspecified. *Inside uncertainty* captures analogous considerations on the side of economic agents who populate the model, a topic we tackle in Section 6. Both types of concerns constitute departures from the rational expectations framework outlined in Section 4, and expose economic agents and the econometrician to additional challenges that they need to cope with.

An extensive part of Lars's research on outside uncertainty deals with misspecification of asset pricing models. The construction of the Hansen–Jagannathan bound described in Section 2 deals with such misspecification from the perspective of theoretical model building. The bound discards a large set of models whose SDF is not sufficiently volatile to rationalize a given set of average returns. In subsequent work, Lars and his coauthors pursued this line of research further by focusing on measures quantifying the extent of model misspecification implied by failures to satisfy the bound, explicitly accounting for statistical uncertainty.

In the study of linear factor models, a common measure of misspecification are the pricing errors associated with valuation of a vector of returns (Shanken 1987, Gibbons et al. 1989). These pricing errors emerge from Euler equations (3), and can be collected as moment conditions to form a GMM estimator. The usual χ^2 statistic from such a GMM test of overidentifying restrictions is informative about the degree of *statistical* rejection of the pricing model specified under the null hypothesis, yet it is harder to provide an economic interpretation of the statistic. For example, more volatile modeled SDFs generate higher statistical uncertainty which reduces the χ^2 statistic, providing weaker statistical evidence against the null hypothesis of a correctly specified model. However, such a lower test statistic does not imply that the model is closer to the true SDF in an economic sense.

Hansen et al. (1995) and Hansen and Jagannathan (1997) consider a space of payoffs and their prices together with a proxy that serves as a candidate SDF. They propose to use the maximum pricing error the proxy generates on this payoffs space and show that one can identify it with the least squares distance of the proxy from the set of all SDFs that correctly price the chosen payoffs. The authors then extend their results to restrict attention to only strictly positive SDFs, and to account for market imperfections like short-sale constraints and transaction costs. More recently, Nagel and Singleton (2011), Gagliardini and Ronechetti (2020), or Antoine et al. (2020) worked on incorporating conditioning information, while Kozak et al. (2020), Feng et al. (2020) coped with disciplining the proliferation of large numbers of new factors used as proxies for the true variability in the SDF. This work brings to the forefront conceptual challenges related to efficient choice of moment conditions from Hansen (1985) and Hansen and Singleton (1991, 1996).

The analysis of SDF misspecification in asset pricing models can be refined by focusing attention on specific components in the martingale decomposition (6). We now consider Euler equations of the form

$$\mathbb{E}_t\left[\kappa_{t+1} m_{t+1} R_{t+1}^i\right] = 1 \tag{9}$$

where m_{t+1} is a reference SDF model imposed by a particular theory of interest, and κ_{t+1} is a distortion that is required to make the set of Euler equations hold. Under incomplete markets, there are typically infinitely many choices of κ_{t+1} that satisfy the set of equations (9) but if these equations do not hold for $\kappa_{t+1} \equiv 1$, then the SDF m_{t+1} is misspecified. The goal is then to choose a κ_{t+1} that minimizes a particular measure of dispersion like entropy, subject to constraints (9). This yields an answer to the question what is the minimal perturbation of the reference SDF needed to make the model consistent with observed returns.

Alternative specifications of κ_{t+1} lead to different economic interpretations. Bansal and Lehmann (1997), Hansen and Renault (2010), Hansen (2012b) and Hansen and Scheinkman (2017) argue that a range of distortions with appealing economic interpretations take the stationary form $\kappa_{t+1} = k\left(x_{t+1}\right)/k\left(x_t\right)$, which implies a transient misspecification of the SDF, leaving the martingale component intact. On the other hand, in the analysis of misspecification of the recovery approach in Borovička et al. (2016), the reference SDF m_{t+1} that can be identified using cross-sectional asset price data is restricted to be transitory, while the misspecification κ_{t+1} forms a martingale.

When κ_{t+1} is restricted to be an increment to a martingale, then it can also be equivalently interpreted as a belief distortion of investors' subjective beliefs relative to the data-generating measure. Ghosh and Roussellet (2019) and Chen et al. (2020, 2021) study identification and estimation of such belief distortions. Such analyses can also be further enriched by the increasing availability of rich survey data on beliefs of investors, households and professional forecasters. Piazzesi et al. (2015), Bhandari et al. (2019), and Szöke (2021) combine these survey data with theoretical restrictions on belief distortions to study quantitative implications of subjective beliefs for asset prices and macroeconomic dynamics.

A related research agenda concerns analysis of time-series models with misspecifications that may be particularly prominent at specific frequencies. Hansen and Sargent (1993) analyze the use of seasonally adjusted and unadjusted data for the study of business cycle fluctuations. In theory, we would like to write down and estimate an economic model with the correct structural interpretation of business cycles and seasonality, even if we are interested only in model fit at business-cycle frequencies, because of potential cross-equation restrictions that relate both types of fluctuations (Sargent 1978a). However, imposing a misspecified model of seasonality may contaminate estimates of parameters relevant for the business cycle, in which case it may be preferable to estimate the model without an explict seasonal component but with data adjusted for seasonality using a statistical filter. Which of the two approaches yields better estimates depends on the particular model of the economy, and Hansen and Sargent (1993) show that in a range of plausible applications, the use of seasonally adjusted data is preferable. With the correct model of seasonality, the use of seasonally adjusted data does not lead to a significant loss in efficiency, while with a misspecified seasonal component, the biases in the estimates may be substantial.

These considerations also had an impact on developing strategies for estimation of misspecified models of continuous-time diffusions. Hansen and Scheinkman (1995) laid foundations for this work by constructing moment conditions for identification and GMM estimation of these diffusions derived from their infinitesimal generators. Their subsequent research shaped these moment conditions in ways that were meant to guard the econometrician against potential misspecifications of high frequency dynamics. Much of this research is summarized in the handbook chapter by Aït-Sahalia et al. (2010), published in the Handbook of Financial Econometrics that Yacine Aït-Sahalia and Lars edited.

Conley et al. (1997b) apply this strategy to estimate diffusion models for the Federal funds rate. This extremely popular class of models (Vasicek 1977, Cox et al. 1985) has been used as a building block for term structure modeling and derivative pricing but functional forms of the drift and volatility components are often used for analytical tractability, rather than being carefully estimated. Since drift parameters are inherently hard to estimate from local behavior of a diffusion, the authors condition on a constant volatility elasticity and estimate a flexible nonlinear drift function using moment conditions that are based on the stationary distribution of interest rate data, thus exploiting long-run properties of the underlying diffusion. The volatility elasticity is then estimated using local evolution over adjacent observations. Conley et al. (1997a) provide bootstrap methods to implement the long-run conditions in a tractable way.

Continuing the work on long-restrictions used to identify parameters of diffusion processes, Hansen et al. (1998) propose an identification strategy based on the stationary distribution and the most persistent component of the process. Focusing attention on persistent components is an appealing identification method in situations when misspecifications at higher frequencies occur, for example due to contamination of data with high-frequency noise, such as seasonal effects. Chen et al. (2009) solve a closely related problem and extract nonlinear principal components of multivariate diffusions that penalize short-run variability and hence focus on maximizing long-run variation. Chen et al. (2010) study mixing conditions implied by coefficients of diffusions processes that determine the decay rates of the impact current shocks have on long-horizon evolution of the diffusion.

6 Model misspecification concerns faced by economic agents

The research described in the preceding section is explicitly concerned with analysis of model misspecification from the perspective of the econometrician. But if the econometrician struggles to pin down the structure of the 'true' model, it is plausible to assume that agents in the model face similar challenges. This desire to put the struggling econometrician and agents inside the model on a similar footing led Lars and Thomas Sargent to pursue a path of research focused on modeling economic agents as decision makers who face not only risk characterized by well-defined probability distributions but who must also cope with uncertainty that does not have a probabilistic structure.

Over time, this work has evolved from adapting and extending results from linear-quadratic optimal control to modeling agents who contemplate alternative parameterized theories as approximations of the real world, and surround these theories with unstructured model uncertainty. In these most recent contributions, the decision makers inside the model engage in scientific reasoning that in many aspects resembles the work of an actual econometrician.

Many of the results are described in detail in the comprehensive 'Robustness' book (Hansen and Sargent 2008), and a more concise and applied summary can be found in the Hansen and Sargent (2011b) chapter written for the Handbook of Monetary Economics. Lars's 2007 Ely Lecture (Hansen 2007) elucidates challenges faced by economic agents when coping with uncertainty from a broader perspective, making connections to learning problems, asset valuation, as well as to the econometrician's concerns about model misspecification described in Section 5.

The research agenda can be traced back to Hansen and Sargent (1995), who followed Jacobson (1973, 1977) and Whittle (1981, 1989, 1990) and extended the dynamic linear-quadratic-Gaussian optimal regulator model by introducing a *risk-sensitive* adjustment that increases effective risk aversion relative to the quadratic cost function. By suitably structuring the functional form of the adjustment, Hansen and Sargent (1995) achieve a formulation that preserves time-invariant decision rules in the infinite-horizon limit, without the effects of this risk-sensitive adjustment vanishing over time as in the earlier literature. At the same time, the formulation preserves tractability of the linear-quadratic framework with quadratic value functions a linear optimal decision rules. In particular, the continuation value recursion (7) in the linear-quadratic problem is modified as

$$V\left(x_t\right) = \max_{u_t} v\left(x_t, u_t\right) - \frac{\beta}{\sigma} \log \mathbb{E}_t \left[\exp\left(-\sigma V\left(x_{t+1}\right)\right)\right], \tag{10}$$

where $\sigma > 0$ is a risk-sensitivity parameter, with the regulated law of motion given by (8). Taking the limit $\sigma \to 0$ yields the original recursion (7).

The original linear quadratic problem (7)–(8) exhibits a feature known as *certainty equivalence*—the optimal decision rule $u_t = Fx_t$ does not depend on the matrix C that controls the volatility of shocks perturbing the law of motion for x_t. In the risk-sensitive problem, on the other hand, the matrix F in the optimal decision rule will depend on both C and σ. This is the notion in which the problem involves an additional degree of aversion to risk, governed by the parameter σ.

Anderson et al. (1998) and Hansen et al. (1999) provide a reinterpretation of recursion (10) in the context of a *robust* decision maker. In the rational expectations framework (7), agent's subjective probability measure that determines the expectations operator $\mathbb{E}_t$ coincides with the objective data-generating measure. A robust decision maker is instead concerned that this probability measure, or model, may be misspecified, and therefore views it only as an approximation of the data-generating process. Models that are statistically hard to distinguish from this approximating model are viewed as plausible candidates for the data-generating model, and the decision maker desires to devise decision rules that do not underperform under any of these candidate models.

Formally, the continuation value recursion of the robust decision maker can be written as a minimax problem

$$V(x_t) = \min_{\substack{h_{t+1} \\ \mathbb{E}_t[h_{t+1}]=1}} \max_{u_t} v(x_t, u_t) + \beta \mathbb{E}_t [h_{t+1} V(x_{t+1})] + \frac{\beta}{\sigma} \mathbb{E}_t [h_{t+1} \log h_{t+1}] . \tag{11}$$

Each candidate model is indexed by a strictly positive random variable h_{t+1} with conditional mean equal to one. Formally, h_{t+1} is the change of measure between the candidate and approximating model, and can be understood as a weighting scheme over next-period outcomes. The last term in (11) is the entropy of h_{t+1}, which penalizes statistically large deviations from the approximating model. The penalty is non-negative, and zero for $h_{t+1} \equiv 1$, which corresponds to choosing the approximating model. The parameter σ now controls the magnitude of the entropy penalty—a large value of σ implies a small penalty for statistical deviations, and the decision maker effectively explores a larger set of models as statistically plausible. The minimization problem over h_{t+1} leads to the choice

$$h_{t+1} = \frac{\exp(-\sigma V(x_{t+1}))}{\mathbb{E}_t [\exp(-\sigma V(x_{t+1}))]}. \tag{12}$$

The agent therefore chooses a probability distribution, called the *worst-case* model, that is exponentially tilted, relative to the approximating model, toward next-period states with low continuation values $V(x_{t+1})$. Just as in the rational expectations framework from Section 4, this worst-case model is an endogenous outcome determined in equilibrium, but now it is explicitly distinct from the data-generating measure.

The minimax formulation is the formalization of the desire to choose controls u_t in a utility maximizing way, but taking into account the performance of the decision rule under adversely slanted models. Since the continuation value $V(x_{t+1})$ is co-determined by the choice of the optimal control, the worst-case model and the decision rule interact.

The set of models that the decision maker explores as plausible alternatives is unstructured and large, only limited by the entropy penalty the decision maker incurs, and the decision maker does not impose any prior over it. The problem can therefore be viewed as a particular model of decision-making under Knightian uncertainty (Knight 1921). Later, Strzalecki (2011) provided an axiomatic foundation of choice behavior under ambiguity aversion that leads to representation (11).

In this way, the robust preference model falls into the category of models of ambiguity aversion that include, for example, multiple-prior preferences (Gilboa and Schmeidler (1989), Epstein and Schneider (2003)) or smooth ambiguity averse preferences (Klibanoff et al. 2005, 2009). Hansen and Sargent (2001b) outline these connections more formally. But while tightly linked in terms of preference representations, the original motivation of these two literatures had been quite distinct. Gilboa and Schmeidler (1989) and follow-up papers investigated alternative axiomatizations of preferences that would address some of the empirical inconsistencies of the subjective expected utility framework of Savage (1954), in particular the Ellsberg (1961) paradox. On the other hand, Hansen and Sargent strived to robustify decision rules in optimal control theory against potential model misspecification, combining tools from engineering, statistical decision theory and robust Bayesian methods from econometrics (Clement and Hansen 2015). The unifying feature is the exposure of the decision maker to nature that chooses adversely tilted probability distributions, leading to the minimax form of the decision problem.

Substituting the distortion (12) back into recursion (11) yields a representation that is identical to the risk-sensitive recursion (10). As Anderson et al. (1998) and Hansen et al. (1999) note, this reveals that the robust decision-making model is, from the perspective of risky choices, observationally equivalent to risk-sensitive preferences, or, in the case of logarithmic period utility, to Epstein and Zin (1989, 1991) preferences with unitary elasticity of intertemporal substitution and relative risk aversion equal to $1 + \sigma$.

In the asset pricing context, the pricing restriction (3) of a robust decision maker becomes

$$\mathbb{E}_t^h \left[m_{t+1} R_{t+1}^i \right] = \mathbb{E}_t \left[h_{t+1} m_{t+1} R_{t+1}^i \right] = 1,$$

where $\mathbb{E}_t^h$ is the expectation operator under the worst-case model determined by (12), and m_{t+1} is the standard marginal rate of substitution derived from the period utility function $v\left(x_t, u_t\right)$. The first equality comes from the change-of-measure interpretation of h_{t+1}. The Euler equation of a robust decision maker acting under the worst-case model can be equivalently interpreted as an optimality condition of a rational investor with a distorted SDF $m_{t+1}^h \doteq h_{t+1} m_{t+1}$, a special case of equation (9). In both cases, h_{t+1} depends on the parameter σ.

The distinction comes from the structural interpretation of the parameter σ. Under the recursive utility model of Epstein and Zin (1989) or risk-sensitive model of Hansen and Sargent (1995), σ measures risk aversion with respect to gambles with well-understood probability distributions. Weil (1989) and Tallarini (2000) note that models with these preferences can generate risk premia comparable to those observed in financial markets, but only when σ is calibrated to risk aversion that is implausibly high. On the other hand, under the robust interpretation, σ controls the statistical distance between the approximating and worst-case model.

Anderson et al. (2000) and Hansen et al. (2002) advocate the use of detection error probabilities to calibrate σ. The detection error probability is the average of type I and type II errors in the likelihood ratio test between the approximating and the worst-case model for a given sample length, and thus reflects the decision maker's lack of confidence in the ability to distinguish the two models using available data. While there is no agreed upon threshold for the detection error probability calibration, Hansen et al. (2002) and Cagetti et al. (2002) argue that thresholds with meaningful degrees of model uncertainty amplify required risk compensations in quantitatively substantial ways and bring risk premia in conventional models close to observed data. Ju and Miao (2012) and Collard et al. (2018) provide other quantitative assessments of the role of ambiguity aversion in the context of financial markets.

While these risk premia are in both cases manifested in observable data in the form of differences in average returns, their structural interpretation differs. Recall from (4) that the expected excess return on asset i is determined by the covariance with the SDF, $-R_t^f Cov_t \left(m_{t+1}^h, R_{t+1}^i \right)$. Given that

$m_{t+1}^h \doteq h_{t+1} m_{t+1}$, the expected excess return involves a component that reflects compensation for risk, m_{t+1}, and a component that reflects compensation for Knightian uncertainty, h_{t+1}. The delineation is particulary sharply characterized in Anderson et al. (2003) for the continuous-time environment. In that paper, the authors describe three drift and jump distortions of the data-generating process that reflect adjustments for risk and uncertainty, and the local evolution of a bound on the detection errors. In doing so, they are able to provide a precise link between uncertainty compensations and detection error probabilities.

Even though this distinction between compensation for risk and uncertainty has no bearing on observed asset prices, policy implications differ. Barillas et al. (2009) follow the methodology introduced by Hansen et al. (1999) and Alvarez and Jermann (2004) and use prices of risky assets to determine welfare costs of business cycle fluctuations. Since the risk premium on an asset that pays the aggregate consumption stream is high, investors require high compensations for holding this risk, and would, reciprocally, be willing to pay large amounts to remove it. While in the risk-based story, there are large welfare gains from removing aggregate consumption risk, most of the welfare gains in Barillas et al. (2009) can be achieved by removing model uncertainty.

Hansen, Sargent, and their coauthors also extensively worked on deeper theoretical aspects underlying model misspecification concerns. Preferences specified by the recursion (11) involve an entropy penalty with an exogenous penalty parameter σ, and the entropy of the worst-case model is an endogenous outcome. However, β/σ can also be interpreted as a Lagrange multiplier on an entropy constraint $\mathbb{E}_t\left[h_{t+1}\log h_{t+1}\right] \leq \eta$, where η is the exogenous parameter and σ is determined endogenously. Hansen and Sargent (2001b) call the former specification *multiplier* preferences, and the latter *constraint* preferences.

It is typically possible to find a state-dependent calibration of one parameter to make the outcomes ex-post identical to the other specification but implications for counterfactual exercises will differ. As Hansen et al. (2006) lucidly explain, the multiplier preference model closely connects the robust control model to risk-sensitive preferences, while the constraint preference model naturally falls into the multiple prior framework of Gilboa and Schmeidler (1989) and Epstein and Schneider (2003). Maccheroni et al. (2006a,b) provide an axiomatic foundation of ambiguity averse preferences that extend multiplier preferences with more general penalty functions.

While the papers cited above use both discrete- and continuous-time specifications, providing a tight link between the two environments is delicate. Skiadas (2013) provides limiting arguments under which ambiguity concerns vanish for many specifications of ambiguity averse preferences and risk structures as the length of the time period converges to zero. Hansen and Sargent (2011a) and Hansen and Miao (2018) analyze these limiting arguments in more detail and offer alternative ways how to think about preserving the role of ambiguity as frequency of arrival of information increases. Similar considerations are also reflected in the small-noise approximations of robust preferences in Anderson et al. (2012) and Borovička and Hansen (2014b).

The continuation value in (11) can also be viewed as the value function for a zero-sum game between a benevolent player who chooses optimal control u_t and a malevolent player who chooses an adverse probability distribution given by the distortion h_{t+1} (Hansen and Sargent 2001a). The solution to the problem then depends on the set of strategies which the players are allowed to use. Hansen and Sargent (2005, 2007) study two cases in the presence of hidden state variables, one in which the players are equipped with a commitment technology, and one without commitment. They formulate corresponding time-0 sequence problems for the game and show conditions under which these sequence problems can be recursified to obtain problem (11).

The presence of hidden state variables also allows for interaction of pessimistic belief biases with learning. The need to estimate hidden states increases uncertainty in the model, and time variation in this uncertainty generates fluctuation in valuations. Since belief distortions (12) of robust agents depend on continuation values, and these continuation values vary with changes in uncertainty, this feedback effect can generate 'belief fragility' in the form of large fluctuations in the pessimistically slanted worst-case probability measure that can have a pronounced effect on asset prices. Hansen and Sargent (2010) study a quantitative calibrated economy with robust agents who are uncertain about the current state of the economy as well as the approximating model, and show that these feedback effects can be quantitatively substantial. Hansen et al. (2010) study a similar learning problem in a linear-quadratic-Gaussian setup.

Engaging agents inside the model in learning problems brings their decision problems closer to those of the outside econometricians, who frequently use filtering methods in estimation problems constrained by data availability. In their most recent work, Hansen and Sargent also substantially elaborated the structure of the environment to allow for more sophisticated reasoning of the decision makers in the presence of uncertainty.

The robust preference specification characterized by the continuation value recursion (11) requires the calibration of a single parameter σ that determines the decision maker's attitudes to model misspecification, and the decision maker treats all uncertainty symmetrically. While this approach meets Robert Lucas' dictum of being '*hostile toward theorists bearing free parameters*' (Lucas 1980, p.709), there are plausible situations in which we may believe that the decision maker is more confident about some aspects of the model than others. For example, Hansen et al. (2020) achieve this by allowing the model builder to shape desirably the entropy bound in the constraint preference version of (11).

Substantially more subtlety can be incorporated in the decision problem by distinguishing between structured and unstructured uncertainty. The set of models constrained by the entropy constraint is *unstructured* in the sense that it contains all models that are statistically close to the approximating model, even though they may not have a particular parameteric structure. Because this set is not constrained by parameterizations, it is by design extremely large. On the other hand, the confidence in specific features of the model could be captured by specifying a particular, much smaller set of *structured* models that can differ, for example, in their parameteric specifications.

Hansen and Sargent (2020b) and Cerreia-Vioglio et al. (2020) devise and axiomatize a theoretical framework that distinguishes between these two concepts carefully, and in which the decision maker copes with uncertainty about both. They envision the set of structured models as a set of alternative scientific theories expressed in the form of parameterized models that the decision maker considers as plausible. Uncertainty about which of these alternative theories is valid reflects *model misspecification* concerns. At the same time, the decision maker also understands that each of these theories is potentially only a simplified version of the true data generating process, which is unlikely to be captured by a particular parameterized model. These *ambiguity* concerns lead the decision maker to surround the set of structured models with a larger set of unstructured alternatives that are statistically hard to distinguish using available data, as captured by an imposed entropy constraint.

Aversion to both model misspecification and ambiguity leads the agent to choose minimax decision rules that generalize those from decision problem (11). From the modeling perspective, the challenge dealt with in the above work is the formalization of the problem in a way that the resulting decision rules are both admissible in the sense of Good (1952) or Ferguson (1967), and dynamically consistent. Hansen and Sargent (2020a) then provide quantitative applications of this framework in the asset pricing context. This latest work is very much ongoing research in progress but it shows how much the research agenda developed, starting from simple risk-sensitive adjustments to decision rules in Hansen and Sargent (1995), to developing a structure for decision-making that allows economic agents in the model to engage in scientific reasoning similar to that of model builders and econometricians standing outside the model.

7 Policy-oriented work

The concepts of inside and outside uncertainty described in Hansen (2014) naturally extend to theoretical policy problems and implementation of government policies, and global events of the last decades emphasized the policy relevance of incorporating such model misspecification concerns into economic modeling. First, the government may want to acknowledge that its economic policies are being interpreted by private sector agents who distrust both the economic environment as well as the policies themselves. At the same time, policymakers may also want to incorporate their own doubts about the specification of models they use to design these policies. Lars's recent work contributes both to theoretical analysis of such problems as well as toward applied policy making.

On the theoretical level, interaction of a benevolent government with an optimizing private sector in the presence of model uncertainty leads to novel considerations, both positive and normative. Does the planner or policy maker face the same model misspecification concerns as the private sector, or do these differ? And, on the normative side, *should* the policy maker adopt a benevolent welfare function that takes into account model uncertainty on the side of the private sector, or should the welfare function be based on a paternalistic rational expectations evaluation of the private sector utilities? These considerations lead to interactions between endogenously determined subjective beliefs of the policy maker and the private sector. Departing from Woodford (2010), these questions are explored in Hansen and Sargent (2012) and Hansen and Sargent (2015). Similar issues appear in strategic environments without perfect competition, as in the Stackelberg leader game with robust decision makers studied in Hansen and Sargent (2003).

The aforementioned policy problems are deliberately abstract. In a more concrete policy application, Cogley et al. (2008) study monetary policy experimentation when the monetary authority is uncertain about the true model of the economy and considers two alternatives, one that exhibits a tradeoff between expected inflation and unemployment that the policy maker may exploit, and another where such a tradeoff does not exist. In this environment, distrust in each of the two models encourages policy maker's experimentation, while the distrust in the prior over the two models inhibits it. The paper however abstracts from formally modeling a forward-looking private sector and so the interactions between the policy maker's and private sector beliefs outlined above are absent.

Lars also extensively worked on applying these theoretical concepts to two broad areas of current policy interest in which lack of understanding of the structure of the environment is an inherent feature of the problem. The first line of inquiry concerns government regulation of the financial sector. The Great Recession of 2007-2009 and the associated financial crisis exposed shortcomings of existing relatively simple models used by policymakers, and at the same time lead to passing large amounts of complex regulation to deal with so-called systemic risk. Hansen (2012a) describes the tension between model uncertainty and policy complexity, and speculates to which extent should regulation in an uncertain environment emphasize simpler rules. Given a range of open questions in this area, he calls for more theoretical research on designing policies in such uncertain environments as well as better measurement of policy-relevant quantities guided by theory.

It is also important to emphasize that Lars's involvement in research on financial modeling and regulation has not been limited to writing academic texts. Perhaps even more importantly, he and Andrew Lo direct the Macro Financial Modeling Project started at the Becker Friedman Institute in 2012, which aims at improving theoretical, empirical, and policy-oriented work on macroeconomic and financial linkages. This project brings together academic researchers, policymakers and industry leaders to exchange ideas on research and policy, and supported numerous students in their work on macro-finance modeling.

A second line of inquiry concerns modeling uncertainty in environments that describe the interaction of economic activity and climate change. The key question for policymakers is the tradeoff emerging from weighing future cost of damages associated with climate change against the cost of mitigation in the presence of substantial uncertainty in climate models, political uncertainty in policy implementation, and uncertainty about mitigation efficacy and arrival of new technologies.

Brock and Hansen (2018) argue that binding together climate models with a model of the economy currently involves substantial simplifications, which further increases the need to incorporate relevant model misspecification concerns. Since asset prices observed in financial markets can be useful in assessing future cost of damages as perceived by market participants, Barnett et al. (2020b) study asset pricing implications in a model where agents cope with model uncertainty about both the economy as well as climate change, and show how these two types of uncertainty amplify the overall model misspecification concerns. Barnett et al. (2020a) then study a policy problem in this environment to provide guidance for the optimal allocation of resources for dealing with climate change. A central message emerging from this body of work is that climate change uncertainty does not in any way justify passive policies that postpone policy action. On the contrary, robust rules may well justify more aggressive action today in order to prevent potential serious future outcomes if the true model of the economy and climate is particularly adverse. Sargent (1999) provides an example of such a result in the context of monetary policy.

While these are two specific areas where uncertainty is particularly pronounced, there is no doubt that future development will bring new challenges for policy making that existing policies do not account for. The unfolding coronavirus (COVID-19) pandemic is a stark example (Hansen (2020a,b), Berger et al. (2020)). Accounting for and coping with such uncertainties should be at the forefront of policy research.

8 A research program in the spirit of the Chicago Tradition

> Chicago is a very intense environment, I tell lots of people that I don't go to work looking for compliments because we are very critical of each other's work and so. But there is a notion that economics is to be taken very seriously and it's to be addressing important problems. It's supposed to be rigorous and at same time relevant. That intensity, I think, has been very important for that environment and it dominates lots of interactions and I think it's really been part of the reason Chicago as an economics department or as an economics community has been so productive.
>
> Lars Peter Hansen, Economic Sciences Prize Committee (2013b)

Lars's lifetime work, from his first published note (Hansen et al. 1978) to his most recent research, epitomizes the best of the Chicago Tradition, emphasizing academic rigor and connection between theory and empirical verification (Hansen and Heckman 1996, Browning et al. 1999). Milton Friedman (Friedman 1974) wrote that ' *"Chicago" stands for an approach that takes seriously the use of economic theory as a tool for analyzing a startingly wide range of concrete problems, rather than as an abstract mathematical structure of great beauty but little power; for an approach that insists on the empirical testing of theoretical generalizations and that rejects alike facts without theory and theory without facts.*' Hansen and Sargent (2015) adhered to this tenet when they revisited Friedman's classic paper on optimal policy under uncertainty (Friedman 1953) and exposed the policy maker to substantially more sophisticated decision problems.

Lars's research connects diverse methodological approaches represented at Chicago, from work that formalizes the concept of uncertainty developed by Frank Knight (Knight 1921), and questions the limits of our understanding of the economy (Hayek 1974), over mathematical, quantitative, and empirical methods embodied in the project of the Cowles Commission, to studying overidentifying restrictions of rational expectations models in finance and macroeconomics, with seminal contributions by Robert Lucas.

The versatility of his work has also been reflected in his teaching and student advising. The first-year Ph.D. econometrics course that he teaches has traditionally been a blend of econometrics, decision theory, macroeconomics and finance. He served as the chairman of the Ph.D. dissertation committee for more than sixty students who now work in academia, policy institutions, and the private sector, including former president of the Federal Reserve Bank of Minneapolis Narayana Kocherlakota, and current governor of the Bank of Israel Amir Yaron. His students are also frequent coauthors—a range of contributions discussed in this essay was written with Ravi Jagannathan, William Roberds, Masao Ogaki, John Heaton, Erzo Luttmer, Timothy Conley, Amir Yaron, Thomas Tallarini, Evan Anderson, Marco Cagetti, Junghoon Lee, Michael Barnett and myself. Scores of graduate students in finance learned from John Cochrane's *Asset Pricing* textbook (Cochrane 2005), heavily influenced by Lars's work on asset pricing methodology.

The southern façade of the Social Science Research Building at the University of Chicago features an abbreviated version of William Thomson's (Lord Kelvin's) quote chiseled underneath one of the windows: "*When you cannot measure your knowledge is meager and unsatisfactory.*" (Thomson 1883, p.73). Merton et al. (1984) provide a fascinating account of the history behind this quote.

Frank Knight famously sneered at the quote when applied to his own field: "*in [economics], the Kelvin dictum very largely means in practice, 'if you cannot measure, measure anyhow!'*" (Knight 1940). Knight recognized the importance of mathematics as an organizing device or common language of economics but was both bitterly against economists who use mathematical models as an oversimplification of the world as well as a vocal skeptic regarding the possibility of turning economics into a quantitative scientific field. I dare to say that Lars's research agenda, carefully designing econometric methods and economic models to account for the uncertainties that both economists outside the model and economic agents inside the model are facing, would have changed his mind in a notably more sympathetic direction.

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